The Lattice of Compactifications of a Locally Compact Space

By

Luke A. Eichelberger

Submitted to the Department of Mathematics and the
Graduate Faculty of the University of Kansas
in partial fulfillment of the requirements for the degree of
Master of Arts

Jack Porter, Chairperson

Committee members

Estela Gavosto

Daniel Katz

Date defended: 5-13-2016

The Master's Thesis Committee for Luke A. Eichelberger certifies
that this is the approved version of the following master's thesis :

The Lattice of Compactifications of a Locally Compact Space

Jack Porter, Chairperson

Date approved: 5-13-2016

Abstract

This is an expanded version of [5] by Magill. The results of [5] are proven with greater detail and any result stated in [5] but not proven is proven here. Let $\mathscr{K}(X)$ and $\mathscr{K}(Y)$ be used to indicate the lattice of Hausdorff compactifications of locally compact, non-compact spaces X and Y with X and Y Tychonoff. This paper primarily concerns how a lattice isomorphism between $\mathscr{K}(X)$ and $\mathscr{K}(Y)$ exists if and only if a homeomorphism between particular extensions of X and Y exists with specified properties. On the way to proving the main results, we prove several lemmas about $\beta - families$ of compact extensions of Tychonoff spaces. Some of the Lemmas slightly generalize corresponding lemmas in [5]. Efforts are made to make this paper self-contained.

Acknowledgements

I would like to thank my advisor Professor Jack Porter for showing me how interesting topology can be, and for eagerly agreeing to work with me when I wanted to deepen my knowledge of the subject. His abundance of help, guidance, and support made working on this project feasible and rewarding.

I would also like to thank the members of my committee: Professor Estela Gavosto, who reminds me that mathematics is about ideas, both big and small, and who has been there for me when I needed extra support; and Professor Daniel Katz, who goes out of his way to help students who seek additional knowledge, and whose high standards have encouraged me in grasping concepts that initially seemed beyond reach.

I wouldn't be where I am without the support of family and friends. The encouragement of Avary Kolasinski, Nick Ma, and John Reynolds helped me to push forward when my doubts were too large to ignore. My family is a source of endless support, and reminds me there is more to life than mathematics.

Finally, I would like to thank Lori Springs, who cares deeply for graduate students, and makes sure to take care of us.

Contents

Chapter 1

Introduction

Hausdorff Compatifications have been studied since the 1920s, if not earlier, when the foundations were established through papers of Alexandroff, Cartan, Čech, Lubben, Stone, Tychonoff, Urysohn, and Wallman between 1929 and 1941 [1]. It was through Čech [1937] that the Stone-Čech compactification used in the present paper was established. While competing notions of compactness were considered in the early 1900s, including what we now call countable compactness and sequential compactness, the modern definition seems to have been accepted by the 1940s. Some impetus for this was Tychonoff's Theorem (published in 1937) that the arbitrary product of compact spaces is compact. This property is not shared by countably compact or sequentially compact spaces. That the modern definition of compactness was “correct" is underscored by a variety of theorems that hold for compact spaces but not spaces in general, such as the Extreme Value Theorem (that every continuous function on a compact space is bounded and attains a maximum value).

In order to prove the results in this paper, we must assume a space is Hausdorff and Completely Regular. Let $\mathscr{K}(X)$ indicate the collection of all Hausdorff compactifications of a Tychonoff space X. We will say that Hausdorff compactifications αX and γX are equivalent (and identified as the same) if there exists a homeomorphism $h : \alpha X \to \gamma X$ such that $h(x) = x$ for all $x \in X$. It is known (but we will show) that $\mathscr{K}(X)$ is not only a collection but a set. We can then define a relation $\leq$

on $\mathscr{K}(X)$ by $\alpha_1 X \leq \alpha_2 X$ iff there exists a continuous onto function from $\alpha_2 X$ to $\alpha_1 X$ that fixes X. This relation is a partial order. In addition, $(\mathscr{K}(X), \leq)$ is a complete lattice if and only if X is locally compact.

Now consider a lattice isomorphism Γ between $\mathscr{K}(X)$ and $\mathscr{K}(Y)$ where X and Y are locally compact but not compact. This paper investigates the relationship between Γ and a homeomorphism between $\beta X \backslash X$ and $\beta Y \backslash Y$ where βX indicates the Stone - Čech compactification of X. In addition, two corollaries are proven concerning the automorphism group $\mathscr{A}(\mathscr{K}(X))$ of the lattice $\mathscr{K}(X)$. The precise statements of these results are given in 4.1 through 4.6. We now discuss the material from general topology used within the paper.

Chapter 2

Preliminaries

2.1 Basic Concepts

We begin by defining the separation axioms. Most separation axioms are, as the name suggests, axioms that allow us to enclose points or sets by disjoint sets that are typically open or closed. That is, we can separate points or sets by other sets.

Note: The word space will be used to mean topological space in all that follows.

Definition 2.1.1. (a) A space X is called T_1 if for distinct $x, y \in X$, $x \notin cl\{y\}$ and $y \notin cl\{x\}$

(b) A space X is called **Hausdorff** if for any two points $x, y \in X$, there exist disjoint open sets $U, V \in \tau(X)$ such that $x \in U$ and $y \in V$.

(c) A space X is called **completely regular** if for each closed set A and $p \in X \backslash A$, there is a continuous function $f : X \to \mathbb{R}$ such that $f(p) = 1$ and $f[A] \subseteq \{0\}$. A completely regular T_1 space is called **Tychonoff**.

(d) A space X is called **normal** if for any pair of disjoint closed sets A and B, there exist disjoint open sets $U \supseteq A$ and $V \supseteq B$.

We now define other terms from topology that are used freely in this paper. In addition, we prove some basic results involving these terms. The results will be cited when used.

Definition 2.1.2. Let X be a space and $A \subseteq X$ with $x \in X$.

(a) The point x is called a **closure point** of A if $U \cap A \neq \emptyset$ for each open set U such that $x \in U$.

(b) The set of all closure points of A is denoted by clA or $cl_X A$ to emphasize the underlying space.

Definition 2.1.3. (a) Let X be a space and $x \in X$. A subset N of X is a **neighborhood (nhb)** of x if there is an open set U such that $x \in U \subseteq N$.

(b) A family $\mathscr{V}_x$ is a **neighborhood base of** x if $\mathscr{V}_x \subseteq \mathscr{N}_x$ and for each $N \in \mathscr{N}_x$, there is $V \in \mathscr{V}_x$ such that $V \subseteq N$.

Definition 2.1.4. Let X be a space, Y a set, and $g : X \to Y$ a function. We say that Y has the **quotient topology induced by** g if U is open in Y if and only if $g^{\leftarrow}[U]$ is open in X. It is easily seen with complements that Y has the quotient topology induced by g if and only if A is closed in Y if and only if $g^{\leftarrow}[A]$ is closed in X.

Definition 2.1.5. Let X be a space. A collection $\mathscr{A}$ of subsets of X is a **cover** of X if $X = \bigcup\{A : A \in \mathscr{A}\}$. An **open cover** is a cover consisting of open sets.

Definition 2.1.6. A space X is said to be compact if every open cover has a finite subcover. That is, if $\mathscr{A}$ is an open cover of X, there exists a subcollection $\{U_1, U_2, ..., U_n\} \subseteq \mathscr{A}$ such that $X \subseteq \bigcup_{i=1}^{n} U_i$

Definition 2.1.7. A space X is said to be **locally compact** if every point of X has a neighborhood base of compact closed neighborhoods.

Definition 2.1.8. A space X is said to be **connected** if X is not the union of two nonempty disjoint open sets. A subspace $A \subseteq X$ is said to be **connected** if A is connected as a subspace of X.

2.2 Compactifications and Extensions

Definition 2.2.1. A space Y is an **extension** of a space X if X is a dense subspace of Y (Recall that X is dense in Y if $cl_Y(X) = Y$). Any extension which is Hausdorff is called a **Hausdorff extension**.

Definition 2.2.2. Let $\alpha_1 X$ and $\alpha_2 X$ be two extensions of a space X. We say that $\alpha_1 X$ is **isomorphic** to $\alpha_2 X$ if there is a homeomorphism $f : \alpha_1 X \to \alpha_2 X$ such that $f(x) = x$ for all $x \in X$.

Definition 2.2.3. A compact extension Y of a space X is called a **compactification** of X.

Theorem 2.2.4. *Let f, g: $X \to Y$ be continuous where Y is Hausdorff. If $D \subseteq X$ is dense in X and $f|_D = g|_D$ then $f = g$ on X.*

Proof. We use contradiction. Suppose $x \notin D$ and $f(x) \neq g(x)$. Since Y is Hausdorff, there exist disjoint open sets U, V in Y such that $f(x) \in U$ and $g(x) \in V$. Since f and g are continuous, $U_1 := f^{\leftarrow}[U]$ and $V_1 := g^{\leftarrow}[V]$ are each open sets containing $\{x\}$. Since D is dense in X, there exists $y \in D \cap U_1 \cap V_1$. But then $f(y) = g(y) \in U \cap V$, contradicting that $U \cap V = \emptyset$.

$\square$

Definition 2.2.5. Suppose $\alpha_1 X$ and $\alpha_2 X$ are two extensions of X. We say that $\alpha_1 X \leq \alpha_2 X$ if there exists a continuous, onto function $h : \alpha_2 X \to \alpha_1 X$ such that $f(x) = x$ for all $x \in X$. We will sometimes write $\alpha_2 X \geq \alpha_1 X$ in place of $\alpha_1 X \leq \alpha_2 X$.

Recall that two extensions Y and Z of a space X are said to be (topologically) isomorphic if there exists a homeomorphism between them fixing X (This is Definition 2.2.2). Below, we prove a lemma regarding isomorphism between spaces.

Lemma 2.2.6. *Suppose $\alpha_1 X$ and $\alpha_2 X$ are two Hausdorff extensions of X. Then $\alpha_1 X \leq \alpha_2 X$ and $\alpha_2 X \leq \alpha_1 X$ if and only if $\alpha_1 X$ is isomorphic to $\alpha_2 X$.*

Proof. ($\Rightarrow$) By assumption, there exists continuous onto functions $f : \alpha_2 X \to \alpha_1 X$ and $g : \alpha_1 X \to \alpha_2 X$, both fixed on X. But $(f \circ g)(x) = x$ for all $x \in X$ and likewise for $g \circ f$. By Theorem 2.2.4, $f \circ g = id_{\alpha_1 X}$ and $g \circ f = id_{\alpha_2 X}$. Thus f is invertible with inverse g, which implies that f is bijective and continuous with a continuous inverse, making $\alpha_1 X$ homeomorphic to $\alpha_2 X$.

($\Leftarrow$) If $\alpha_1 X$ is homeomorphic to $\alpha_2 X$ then there exists a continuous bijection $f : \alpha_1 X \to \alpha_2 X$ fixing X and with a continuous inverse $f^{\leftarrow}$. Thus $\alpha_2 X \leq \alpha_1 X$. But then $f^{\leftarrow}$ is a continuous onto function from $\alpha_2 X$ to $\alpha_1 X$ fixing X, implying $\alpha_1 X \leq \alpha_2 X$.

$\square$

Theorem 2.2.7. *(M. Stone, E. Čech) Each Tychonoff Space X has a Hausdorff compactification βX with these properties:*

(a) *if $f : X \to Y$ is continuous where Y is a Tychonoff space, then there is a continuous function $\beta f : \beta X \to \beta Y$ such that $\beta f|_X = f$.*

(b) *if Z is a Hausdorff compactification of X with property (a), then Z is isomorphic to βX.*

(c) *if Y is a Hausdorff compactification of X, then $\beta X \geq Y$, and*

(d) *each $f \in C^*(X)$ has a continuous extension to $f^\beta \in C^*(\beta X)$.*

Proof. A constructive proof is provided in 10.4 of [6].

□

Definition 2.2.8. We denote the Stone-Čech compactification generated through the construction in Theorem 2.2.7 by βX.

Theorem 2.2.9. *(a) If X is a compact space, then any closed set in X is compact.*

(b) If X is Hausdorff, then any compact subset is closed.

Proof. (a) Suppose X is compact and A is closed in X. Let $\{U_\alpha \cap A\}_{\alpha \in I}$ be an open cover of A where U_α is open in X for $\alpha \in I$. Then $\{U_\alpha\}_{\alpha \in I} \cup (X \backslash A)$ is an open cover of X ($X \backslash A$ is open since A is closed). By compactness of X, there exists a finite subcover of X, $\{U_{\alpha_i}\}_{i=1}^n \cup (X \backslash A)$. But since A is contained in X, and A is disjoint from $X \backslash A$, A is contained in $\{U_{\alpha_i}\}_{i=1}^n$. Thus A is contained in $\{U_{\alpha_i} \cap A\}_{i=1}^n$ which implies that A is compact.

(b) We first note that a set U in a space X is open iff for all $x \in U$, there exists a $V \in \tau(X)$ such that $x \in V \subseteq U$. Suppose A is a compact subset of a Hausdorff space X. Fix $x \in X \backslash A$. Since X is Hausdorff, there exist disjoint sets U_y and V_y in $\tau(X)$ such that $y \in U_y$ and $x \in V_y$ for all $y \in A$. Observe that $\{U_y \cap A\}_{y \in A}$ is an open cover of A. Since A is compact, there exists a finite subcover of A,$\{U_{y_i} \cap A\}_{i=1}^n$. Consider $\bigcap_{i=1}^n V_{y_i}$ where V_{y_i} corresponds to U_{y_i} in the natural way where $i = 1, 2, \ldots, n$. Then V is an open set containing x such that $V \cap A = \emptyset$. Since x was arbitrary in $X \backslash A$, this means $X \backslash A$ is open $\Rightarrow X \backslash (X \backslash A) = A$ is closed. □

Theorem 2.2.10. *(a) If Y is a Hausdorff extension of a compact space X, then $Y = X$.*

(b) Any Hausdorff extension of a compact Hausdorff space Z is just Z itself.

Proof. (a) Suppose Y is a Hausdorff extension of a compact space X. Since X is compact in a Hausdorff space Y, X is closed in Y by Theorem 2.2.9 part b. Thus $X = cl_Y(X) = Y$ as needed.

(b) This is just a restatement of (a)

□

Theorem 2.2.11. *(a) Suppose X is Hausdorff. Then any two disjoint compact subsets of X are contained in disjoint open sets.*

(b) If a space X is Hausdorff and compact, then X is normal.

Proof. (a) Let A and B be disjoint compact sets. Fix $x \in A$. Then for all $b \in B$, there exists open sets $U_b \supseteq \{x\}$, and $V_b \supset \{b\}$ where $U_b \cap V_b = \emptyset$. $\{V_b\}_{b \in B}$ is an open covering of B. Since B is compact, $\{V_b\}_{b \in B}$ contains a finite subcover $\{V_{b_i}\}_{i=1}^n$ which corresponds to $\{U_{b_i}\}_{i=1}^n$. Set $V_x = \bigcup_{i=1}^n V_{b_i}$ and $U_x = \bigcap_{i=1}^n U_{b_i}$. Then $B \subseteq V_x$ and $\{x\} \in U_x$ while $V_x \cap U_x = \emptyset$ (If the intersection was nonempty, there would be some nonempty $V_{b_j} \cap U_{b_j}$, a contradiction). Now repeat the process above for each $x \in A$, generating the sets $\{U_x\}_{x \in A}$ and $\{V_x\}_{x \in A}$. Observe that $\{U_x\}_{x \in A}$ is an open covering of A. By the compactness of A, there is a finite subcover $\{U_{x_i}\}_{i=1}^m$. Now form $U = \bigcup_{i=1}^m U_{x_i}$ and $V = \bigcap_{i=1}^m V_{x_i}$. It is clear that $A \subseteq U$ and $B \subseteq V$. Suppose $z \in U \cap V$ (so the intersection is nonempty). Then $z \in U_{x_k} \cap V_{x_k}$ for some $k \in \{1, 2, ..., m\}$ which is impossible by construction. Thus $U \cap V = \emptyset$ and A and B can be separated by disjoint open sets.

(b) Suppose X is Hausdorff and compact. Let A and B be disjoint closed sets in X. By Theorem 2.2.9, A and B are compact. By part (a), A and B are contained by disjoint open sets. By the definition of normality, we conclude that X is normal.

□

Theorem 2.2.12. *Let Y and Z be Hausdorff extensions of a space X and $f : Y \to Z$ a continuous function such that $f(x) = x$ for $x \in X$. Then $f[Y \backslash X] \subseteq Z \backslash X$.*

Proof. Assume towards a contradiction that there exists $y \in Y \backslash X$ such that $f(y) \in X$. Since f is fixed on X, there is an $x \in X$ such that $f(x) = f(y)$ (with $x \neq y$). Applying Hausdorff, we observe there are disjoint open sets $U, V \in \tau(Y)$ such that $x \in U$ and $y \in V$. By the definition of subspace,

$U \cap X$ is open in X. But $U \cap X \subseteq X \subseteq Z$. Again by definition of subspace, there exists W open in Z such that $U \cap X = W \cap X$.

Observe that $f(y) \in X \subseteq W \cap X \subseteq W$. By the continuity of f, there exists a set T open in Y such that $y \in T$ and $f[T] \subseteq W$. Also, $y \in V \cap T \subseteq W$ is open in Y. By the density of X in Y, $V \cap T \cap X$ is nonempty. Suppose $p \in V \cap T \cap X$. Since $f[V \cap T \cap X] \subseteq W \cap X = U \cap X$, $p = f(p) \in U \cap X$. But then $p \in V \cap U$, a contradiction which completes the proof.

□

Definition 2.2.13. Suppose $\mathscr{P}$ is a topological property.

(a) We say that $\mathscr{P}$ is **hereditary** if X has $\mathscr{P}$ and whenever $A \subseteq X$, then A has $\mathscr{P}$.

(b) We say that $\mathscr{P}$ is **expansive** if (X, τ) has $\mathscr{P}$ and whenever σ is a topology on X such that $\tau \subseteq \sigma$, then (X, σ) has $\mathscr{P}$.

Theorem 2.2.14. *The property Hausdorff is hereditary and expansive.*

Proof. First we show hereditary. Suppose X is Hausdorff and $A \subseteq X$. Let $a, b \in A$. Since X is Hausdorff, there exist disjoint open sets $U, V \in \tau(X)$ such that $a \in U$ and $b \in V$. But $U \cap A$ and $V \cap A$ are two disjoint open sets of $\tau(A)$ such that $a \in U \cap A$ and $b \in V \cap A$. Thus A is Hausdorff.

Now we show expansive. Suppose (X, τ) is hausdorff, and σ is a topology on X such that $\tau \subseteq \sigma$. We need to show that (X, σ) is Hausdorff. Let $a, b \in X$. Since (X, τ) is hausdorff, there exist sets $U, V \in \tau(X)$ such that $a \in U$, $b \in V$, and $U \cap V = \emptyset$. But $\tau \subseteq \sigma$ so $U, V \in \tau(\sigma)$ as well. Thus (X, σ) is Hausdorff.

□

Chapter 3

Lemmas

3.1 $\beta - families$

From this point forward, unless mentioned otherwise, every space denoted by X is assumed to be completely regular and Hausdorff. Since Hausdorff implies T_1, and a space that is completely regular and T_1 is called Tychonoff, every space X from this point forward is Tychonoff.

Lemma 3.1.1. *Suppose X is a space (by the remark above, we assume X is Tychonoff) and αX is any Hausdorff compactification of X. Then there exists a unique continuous function f_α mapping βX onto αX which leaves the points of X fixed.*

Proof. First suppose X is compact. Since $X = \alpha X = \beta X$, $f : \beta X \to \alpha X$ defined by $f(x) = x$ is the identity on X, thus is continuous, and is the only function fixing X.

Now suppose X is not compact. Define $f : X \to \alpha X$ by $f(x) = x$ (the identity function on X). Then if $U \in \tau(\alpha X)$, $f^{\leftarrow}[U] = U \cap X$ which is open in X. Thus f is continuous. (alternatively, it is well known that the identity function between compatible topological spaces is continuous). By Theorem 2.2.7, there exists a continuous function $f_\alpha : \beta X \to \beta(\alpha X)$ such that $f_\alpha|_X = f$. By Theorem 2.2.10, $\beta(\alpha X) = \alpha X$. For onto, observe that $f_\alpha[\beta X]$ is compact in αX since the image of a compact set is compact. By Theorem 2.2.9, a compact subset of a Hausdorff space is closed. Thus $f_\alpha[\beta X]$ is closed. By taking closures with respect to αX in $X \subseteq f_\alpha[\beta X] \subseteq \alpha X$, we get

$\alpha X \subseteq f_\alpha[\beta X] \subseteq \alpha X$, and conclude that f_α is onto. Finally, we show that f_α is unique. Suppose there exists $g_\alpha : \beta X \to \alpha X$ such that $g_\alpha|_X = f$. Since βX and αX are extensions of X, X is dense in βX. But $f_\alpha|_X = f = g_\alpha|_X$. Thus by Theorem 2.2.4, we conclude that $f_\alpha = g_\alpha$ and f_α is unique.

□

We will refer to f_α as the $\beta - function$ of αX.

Definition 3.1.2. Suppose X is a space (as a reminder, we note that we are assuming X is Tychonoff) and αX is a Hausdorff compactification of X. We will use the symbol $\mathscr{F}(\alpha X)$ to denote the set $\{f^{\leftarrow}(p) : p \in \alpha X \backslash X\}$ and refer to this set as the $\beta - family$ of αX.

Observe that the $\beta - family$ partitions $\beta X \backslash X$ into nonempty compact subsets. Here is why: Each member A of $\mathscr{F}(\alpha X)$ is closed since $\{p\}$ is closed (αX is Hausdorff so αX is T_1 which implies singleton's are closed) and the inverse image of a closed set under a continuous function is closed. Since members of the $\beta - family$ are closed in the compact space βX, they are compact in βX. Also, if a space B is compact in $Z \supseteq W$ for spaces Z and W, then B is compact in W. Thus $\beta - family$ members are compact in $\beta X \backslash X$. There is no overlap between members of $\mathscr{F}(\alpha X)$ since f_α is a function and the union of all members of $\mathscr{F}(\alpha X)$ is $\beta X \backslash X$ because f_α is onto. It is clear that no member of $\mathscr{F}(X)$ is empty. And finally, Theorem 2.2.12 implies that no element of X is contained in a $\beta - family$ member.

3.2 The Lattice of Compactifications

Definition 3.2.1. Let X be a topological space (as another reminder, we note the assumption that X is completely regular and Hausdorff). We denote the class of all compactifications of X which are Hausdorff by $\mathscr{K}(X)$.

In the next lemma, we show that $\mathscr{K}(X)$ is a set. To do this, we identify a compactification with its $\beta - family$. While this hasn't been proven yet, we will see by the remark after Lemma 3.4.1 that up to isomorphism, a compactification can be identified with its $\beta - family$. This is used below.

Lemma 3.2.2. *Let X be a space. Identify a compactification of X with its $\beta - family$. Then up to isomorphism, the collection $\mathscr{K}(X)$ of all Hausdorff compactifications of a topological space X is a set.*

Proof. Write $\mathscr{K}(X) = \{\{\alpha X\} : \alpha X$ is a Hausdorff compactification of X$\}$. Then for a given αX, $\mathscr{F}(\alpha X) = \{\{K_i\}_{i \in I} \cup \{y : y \notin \bigcup_{i \in I} K_i\}\}$ where K_i is compact in $\beta X \backslash X$. Observe that $\mathscr{F}(\alpha X) \subseteq \mathscr{P}(\beta X \backslash X)$. Define a map from $\mathscr{K}(X)$ to $\mathscr{P}(\mathscr{P}(\beta X \backslash X))$ by $\alpha X \to \mathscr{F}(\alpha X)$. By the remark after Lemma 3.4.1, this map is $1-1$. But this means that the size of $\mathscr{K}(X)$ is no larger than $2^{2^{|\beta X \backslash X|}}$. But since X is a set, so is $\beta X \backslash X$.

□

Lemma 3.2.3. *Let X be a space. The set $\mathscr{K}(X)$ of all Hausdorff compactifications of X is partially ordered by $\leq$ (where compatifications are identified as the same if they are isomorphic)*

Proof. Fix a space X. To see that X is partially ordered, we show the three properties stated in Definition 5.2.1 from the appendix.

Reflexive: Let αX be a Hausdorff compactification of X. Define $f : \alpha X \to \alpha X$ by $f(x) = x$ (the identity function). Then f is continuous, onto and fixes X. Thus $\mathscr{K}(X)$ is reflexive under $\leq$.

Symmetric: This is just one direction of Lemma 2.2.6.

Transitive: Let $\alpha_1 X$, $\alpha_2 X$, and $\alpha_3 X$ be Hausdorff compactifications of X where $\alpha_1 X \leq \alpha_2 X$ and $\alpha_2 X \leq \alpha_3 X$. There exist continuous, onto functions $g : \alpha_2 X \to \alpha_1 X$ and $h : \alpha_3 X \to \alpha_2 X$ such that $g(x) = x$ and $h(x) = x$ for all $x \in X$. Then $g \circ h : \alpha_3 X \to \alpha_1 X$ is onto and continuous (compositions of onto functions are onto and compositions of continuous functions are continuous) and $(g \circ h)(x) = g(h(x)) = g(x) = x$ for all $x \in X$. Thus $\alpha_1 X \leq \alpha_3 X$ and $\mathscr{K}(X)$ is transitive.

Since $(\mathscr{K}(X), \leq)$ is reflexive, symmetric, and transitive, we conclude that $(\mathscr{K}(X), \leq)$ is partially ordered.

□

Note that since X is Tychonoff (Hausdorff and completely regular), $\mathscr{K}(X)$ has a maximal element, βX.

Terminology associated with lattices is used in Lemma 3.3.1 and 3.3.2. This terminology is defined in the appendix.

3.3 Comparing Compactifications

Lemma 3.3.1. *Let X be a space. Then $\mathscr{K}(X)$ is a complete upper semilattice with respect to the partial order $\leq$.*

Proof. Let $S = \{\gamma_a\}_{a\in A}$ be an arbitrary subset of $\mathscr{K}(X)$. To show that $\mathscr{K}(X)$ is a complete upper semilattice, we need to show that $\bigvee\{\gamma_a\}_{a\in A}$ exists and is in $\mathscr{K}(X)$. Recall that an element in the product $\prod_{a\in A}\gamma_a X$ is a function $f : A \to \bigcup_{a\in A}\gamma_a X$ such that $f(a) \in \gamma_a X$. Define a function $e : X \to \prod_{a\in A}\gamma_a X$ by $e(x)(a) = x \in X \subseteq \gamma_a X$. Define $e_a : X \to \gamma_a X$ by $e_a(x) = x$. We will show that e is an embedding. That is, that $e[X]$ is a homeomorphism. We must show three things: (i) each e_a is continuous, (ii) the family $\{e_a\}_{a\in A}$ separates points (that is, if $x, y \in X$ and $x \neq y$, then there is some $a \in A$ such that $e_a(x) \neq e_a(y)$, and (iii) the family $\{e_a\}_{a\in A}$ separates points from closed sets (that is, if C is closed in X, $x \in X\backslash C$, then there is some $a \in A$ such that $f_a(x) \neq cl(f_\alpha[C])$. Then Theorem 3.9 from [6] will imply that the function $e : X \to \prod_{a\in A}\gamma_a X$ defined by $e(x)(a) = e_a(x)$ is an embedding (this matches the above defintion of e). That (i) is true is clear since each e_a is the identity function between topological spaces and is thus continuous. For (ii), if $x \neq y$ then $e_a(x) = x \neq y = e_a(y)$ for all $a \in A$. That is, the family $\{e_a\}_{a\in A}$ separates points. For (iii), let B be a closed set in X. Then $e_a[B] = B$ is closed in $X \subseteq \gamma_a X$ so that $cl_X(e_a[B]) = e_a[B]$. A basic result of topology states that for a space X, if $C \subseteq D \subseteq X$, then $cl_A B = cl_X B \cap A$. Thus $cl_{\gamma_a X}(e_a[B]) \cap X = cl_X(e_a[B]) = e_a[B] = B$. Let $p \in X\backslash B$. Then $e_a(p) = p$ for all $a \in A$. Thus $e_a(p) \in X$. However, since $cl_{\gamma_a X}(e_a[B]) \cap X = e_a[B] = B$, $e_a(p) \notin cl_{\gamma_a X}(e_a[B])$ and e_a separates points from closed sets. Therefore e is an embedding. Since e is an embedding, it makes sense to identify $e(x)$ with x.

Continuing, we define $Y = \prod_{a\in A}\gamma_a X$ and set $\gamma X = cl_Y(e[X])$ where γX is to be $\bigvee\{\gamma_a\}_{a\in A}$. Since we are identifying X with $e[X]$, $cl_Y(e[X]$ is an extension of X. Observe that $cl_Y(e[X])$ is closed. By Tychonoff's Theorem, $\prod_{a\in A}\gamma_a X$ is compact (Tychonoff's Theorem states that the arbitrary product of compact spaces is compact). Since $cl_Y(e[X])$ is a closed subspace of a compact space, it too is compact (see Theorem 2.2.9). Since the arbitrary product of Hausdorff spaces is Hausdorff (See Theorem 5.9(c) in [6]), $\prod_{a\in A}\gamma_a X$, is Hausdorff. Since Hausdorff is hereditary, and $cl_Y(e[X])$ is a subset of the Hausdorff space $\prod_{a\in A}\gamma_a X$, $cl_Y(e[X]))$ is Hausdorff. Therefore, $\gamma X \in \mathscr{K}(X)$.

Let $\gamma_a X$ be an arbitrary compactification of X. To see that $cl_Y(e[X]) = \bigvee\{\gamma_a X\}_{a\in A}$, we need to first show that $\gamma X \geq \gamma_a X$. Consider the projection map $p_a : Y \to \gamma_a X$. The projection map is automatically continuous. With the identification of $e(x)$ with x, $p_a(e(x)) = e(x)(a) = x$ so that p_a is the identity on X. If we can show that p_α is onto, then we will have that $\gamma X \geq \gamma_a X$ by definition. Towards this end, we observe that $X = e[X] \subseteq p_a[\gamma X] = p_a[cl_Y(e[X])] \subseteq cl_{\gamma_a Y}(p_a[e[X]]) \subseteq cl_{\gamma_a X}[X] = \gamma_a X$ where the second $\subseteq$ follows by a characterization of continuity. From this we have $X \subseteq p_a[\gamma X] \subseteq \gamma_a X$. Since γX is compact and p_a is continuous, $p_a[\gamma X]$ is compact. Thus $p_a[\gamma X]$ is closed. Taking closures of $X \subseteq p_a[\gamma X] \subseteq \gamma_a X$ in $\gamma_a X$, we have $\gamma_a X \subseteq p_a[\gamma X] \subseteq \gamma_a X$ so that $p_a[\gamma X] = \gamma_a X$ and p_a is onto. Therefore, p_a is a continuous function from γX onto $\gamma_a X$ fixing X where X is dense in γX and $\gamma_a X$. Therefore, $\gamma X \geq \gamma_a X$.

Now suppose that δX is a Hausdorff compactification of X such that $\delta X \geq \gamma_a X$ for all $a \in A$. We need to show that $\delta X \geq \gamma X$. Define $g : \delta X \to \gamma X$ by $g(y)(a) = f_a(y)$ for all $a \in A$ and $y \in Y$. We show g is continuous. Since Y has the product topology, it suffices to show that $p_a \circ g : \delta X \to \gamma_a X$ is continuous for all $a \in A$ (See By Theorem 3.8(f) in [6]). Observe that $p_a \circ g(y) = g(y)(a) = f_a(y)$ for all $y \in \delta X$. But f_a is continuous so $p_a \circ g(y)$ is continuous as well. Now let $x \in X \subseteq \delta X$. Then $p_a \circ g(x) = f_a(x) = x$. Thus $X \subseteq g[\delta X]$. Then, as above, $X \subseteq g[\delta X] = g[cl_{\delta X}(X)] \subseteq cl_Y(g[X]) = cl_Y(e[X]) = \gamma X$. Thus $X \subseteq g[\delta X] \subseteq \gamma X$. Since g is continuous and δX is compact, $g[\delta X]$ is compact and thus closed in γX. Taking closures with respect to γX in $X \subseteq g[\delta X] \subseteq \gamma X$, we get $\gamma X \subseteq g[\delta X] \subseteq \gamma X$ which implies that $g[\delta X] = \gamma X$. Thus g is a continuous onto function from δX to γX fixed on the dense subset X and $\gamma X \leq \delta X$.

$\square$

Lemma 3.3.2. *$\mathscr{K}(X)$ is a complete lattice if and only if X is locally compact.*

Proof. If X is compact, then $|\mathscr{K}(X)| = 1$, $\mathscr{K}(X)$ is complete, and X is locally compact since compact implies locally compact.

We now consider the non-compact (non-trivial) case. Suppose X is non-compact. For the forward direction, suppose $\mathscr{K}(X)$ is a complete lattice (see the appendix for the definition of complete lattice). Then $\bigwedge \mathscr{K}(X)$ exists and is an element of $\mathscr{K}(X)$. Write $\alpha X = \bigwedge \mathscr{K}(X)$. Towards a contradiction, we suppose that there are two sets $K_1, K_2 \in \mathscr{F}(\alpha X)$, the $\beta - family$ of αX. Write $\alpha_1(X; K_1 \cup K_2)$ to be the compactification guaranteed to exist by Lemma 3.4.6. By the proof of Theorem 3.4.8, $\alpha X \geq \alpha_1(X; K_1 \cup K_2)$. Also, $K_1 \cup K_2 \in \mathscr{F}(\alpha_1(X; K_1 \cup K_2))$ but $K_1 \cup K_2 \notin \mathscr{F}(\alpha X)$. Thus $\alpha X \neq \alpha_1(X; K_1 \cup K_2)$ and $\alpha X > \alpha_1(X; K_1 \cup K_2)$. However, $\alpha X = \bigwedge \mathscr{K}(X)$ so $\alpha X \leq \alpha_1(X; K_1 \cup K_2)$. This is a contradiction which arose by assuming there were two sets in $\mathscr{F}(\alpha X)$. Thus $\mathscr{F}(\alpha X) = \{\beta X \backslash X\}$ which implies that $\beta X \backslash X$ is closed in βX and X is open in βX. But this means X is open and dense in the compact Hausdorff space βX (see Corollary 8.14 in [6]). Therefore X is locally compact.

Conversely, assume X is locally compact. Theorem 10.7(a) in [6] states that a space X has a Hausdorff one-point compactification if and only if X is locally compact, Hausdorff, and not compact. Therefore, X has a one-point compactification αX and $\mathscr{F}(\alpha X) = \{\beta X \backslash X\}$. Thus $\alpha X = \wedge \mathscr{K}(X)$ which is a least element of $\mathscr{K}(X)$. By Lemma 3.3.1, X is a complete upper semi-lattice. But Theorem 5.2.6 states that a complete upper semilattice with a least element is a complete lattice. Therefore $\mathscr{K}(X)$ is complete.

□

In the rest of the paper, we will sometimes use the following fact regarding lattices: if L and K are lattices, and f is a function between them, then f is an order-isomorphism if and only if f is a lattice isomorphism. Definitions of these terms, as well as a proof of the fact (see Theorem 5.2.10), are provided in the appendix.

3.4 Constructing Compactifications

Lemma 3.4.1. *Let $\alpha_1 X$ and $\alpha_2 X$ be two Hausdorff compactifications of X. Then $\alpha_1 X \leq \alpha_2 X$ if and only if each set in $\mathscr{F}(\alpha_2 X)$ is a subset of a set in $\mathscr{F}(\alpha_1 X)$.*

$$\begin{tikzcd}
\beta X \arrow[r, two heads, "f_{\alpha_1}"] \arrow[d, two heads, "f_{\alpha_2}"'] & \alpha_1 X \\
\alpha_2 X \arrow[ur, two heads, "h"'] &
\end{tikzcd}$$

Diagram 1

Proof. ($\Rightarrow$) Suppose $\alpha_1 X \leq \alpha_2 X$. Let f_{α_1} and f_{α_2} be the $\beta - functions$ of $\alpha_1 X$ and $\alpha_2 X$ respectively. By definition, this means there exists a continuous function $h : \alpha_2 X \to \alpha_1 X$ such that $h(x) = x$ for all $x \in X$. Observe that if $x \in X$, $h \circ f_{\alpha_2}(x) = h(x) = x = f_{\alpha_1}(x)$ (since h, f_{α_1}, and f_{α_2} are fixed on X). Since X is dense in each extension, Theorem 2.2.4 implies that $h \circ f_{\alpha_2} = f_{\alpha_1}$, just as in the proof of Lemma 3.1 (Thus diagram 1 commutes). Now let $A \in \mathscr{F}(\alpha_2 X)$. That is, $A = f_{\alpha_2}^{\leftarrow}(\{p\})$ for some $p \in \alpha_2 X \backslash X$. Let $q \in A$ so that $f_{\alpha_2}(q) = p$. Then we have $h(p) = h(f_{\alpha_2}(q)) = f_{\alpha_1}(q)$ which implies $q \in f_{\alpha_1}^{\leftarrow}(h(p))$. But $h(p) \in \alpha_1 X$. Call $h(p) = r$. Then we have that $A \subseteq f_{\alpha_1}^{\leftarrow}(r) \in \mathscr{F}(\alpha_1 X)$ as needed.

($\Leftarrow$) Now suppose that each set in $\mathscr{F}(\alpha_2 X)$ is a subset of a set in $\mathscr{F}(\alpha_1 X)$. As in the forward direction of this proof, let f_{α_1} and f_{α_2} be the $\beta - functions$ of $\alpha_1 X$ and $\alpha_2 X$ respectively. Let $p \in \alpha_2 X \backslash X$. By assumption, there exists a $q \in \alpha_1 X$ such that $f_{\alpha_2}^{\leftarrow}(p) \subseteq f_{\alpha_1}^{\leftarrow}(q)$. Note that q is unique since if $f_{\alpha_1}^{\leftarrow}(q) = f_{\alpha_1}^{\leftarrow}(r)$ for some $r \in \alpha_1 X \backslash X$, the definition of a function would be violated. Define $h : \alpha_2 X \to \alpha_1 X$ by $h(p) = q$ and $h(x) = x$ for all $x \in X$. h is well-defined since q is unique. We will show that h is a continuous, onto function from $\alpha_2 X \to \alpha_1 X$, yielding the desired conclusion by Definition 2.2.5. That h is onto is clear by construction. To show continuous, we first show that $f_{\alpha_1} = h \circ f_{\alpha_2}$. Towards this end, let $r \in \beta X \backslash X$. Then $f_{\alpha_2}(r) \in \alpha_2 X$, so by assumption $f_{\alpha_2}^{\leftarrow}[f_{\alpha_2}(r)] \subseteq f_{\alpha_1}^{\leftarrow}(t)$ for some $t \in \alpha_1 X$. By definition of $h, h[f_{\alpha_2}(r)] = t$. Since the inverse image of an image contains the image, $r \in f_{\alpha_2}^{\leftarrow}[f_{\alpha_2}(r)]$ implies $r \in f_{\alpha_1}^{\leftarrow}(t)$ which implies $f_{\alpha_1}(r) = t$. Thus $h \circ f_{\alpha_2}(r) = f_{\alpha_1}(r)$ for all $r \in \beta X \backslash X$ so that $f_{\alpha_1} = h \circ f_{\alpha_2}$ (Thus diagram 1 commutes). Finally, suppose K is a closed subset of $\alpha_1 X$. By the continuity of f_{α_1}, $f_{\alpha_1}^{\leftarrow}[K]$ is closed in βX and thus compact in βX (by Theorem 2.2.9). By the continuity of f_{α_2}, $f_{\alpha_2}[f_{\alpha_1}^{\leftarrow}(K)]$ is compact in $\alpha_2 X$ and thus closed in $\alpha_2 X$ (Theorem 2.2.9 again). But Theorem 5.1.3 implies that $f_{\alpha_2}(f_{\alpha_1}^{\leftarrow}(K)) = h^{\leftarrow}(K)$. Thus $h^{\leftarrow}(K)$ is closed, making h continuous.

□

Now, we observe the following: Suppose we know the $\beta - family$ for αX. Suppose there is another compactification of X, γX with the same $\beta - family$. Then since $\mathscr{F}(\alpha X) = \mathscr{F}(\beta X)$, Lemma 3.4.1 implies that $\alpha X \leq \gamma X$ and $\gamma X \leq \alpha X$. Thus αX and γX are isomorphic (See Theorem 10.2 from [6]). We conclude that the compactification is unique up to isomorphism. That is, the beta family of a compactification determines a compactification (up to isomorphism).

We now prove a result that allows us to generalize some of the Lemmas in [5].

Lemma 3.4.2. *Suppose X is a locally compact space which is not necessarily completely regular or Hausdorff. Suppose X is a dense subspace of αX where αX is Hausdorff. Then*

1. *$\alpha X \backslash X$ is closed and compact.*
2. *If $K \subseteq \alpha X \backslash X$ and K is closed in αX, then K is compact.*
3. *$\alpha X \backslash X$ is normal.*

Proof. Since X is locally compact, and αX is Hausdorff, Theorem 8.13b from [6] implies that $X = U \cap A$ where U is open in αX and A is closed in αX. Therefore, $X \subseteq A \subseteq \alpha X$. Taking closures with respect to αX, we get $\alpha X \subseteq A \subseteq \alpha X$ where the density of X in αX and that A is closed are used. This implies $A = \alpha X$. Thus $X = U \cap \alpha X = U$ since $X \subseteq \alpha X$. Therefore, $\alpha X \backslash X = X \backslash U$ is closed. This proves (1). Since $\alpha X \backslash X$ is a closed subset of a compact space, Theorem 2.2.9 implies

$\alpha X\backslash X$ is compact. But any closed subset of a compact space is compact. Thus K is compact and (2) is proven. Since Hausdorff is hereditary (Theorem 2.2.14), $\alpha X\backslash X$ is Hausdorff. By part (1), $\alpha X\backslash X$ is compact so by Theorem 2.2.11, $\alpha X\backslash X$ is Normal.

□

Lemma 3.4.3. *Let X be a space which is not necessarily completely regular, and let αX be a Hausdorff compactification of X, and let $K_1, K_2, ..., K_N$ be N mutually disjoint nonempty compact subsets of $\alpha X\backslash X$. Choose N distinct points $q_1, q_2, q_3, ..., q_N$ not in αX, and define a mapping h from αX onto $\gamma X = [\alpha X\backslash \bigcup_{i=1}^{N} K_i] \cup \{q_i : i = 1, ..., N\}$ by $h(p) = p$ for $p \in \alpha X\backslash \bigcup_{i=1}^{N} K_i$ and $h(p) = q_i$ for $p \in K_i$. Let γX have the quotient topology induced by h. Then γX is a (Hausdorff) compactification of X.*

Before proving this lemma, we remark how Lemma 3.4.2 is used to generalize Lemma 2 from [5]. In [5], it is assumed that X is locally compact and $K_1, K_2, ..., K_N$ are mutually disjoint nonempty *closed* subsets of $\alpha X\backslash X$. However, if we assume X is locally compact, Lemma 3.4.2 implies that the K_i's are compact. By relaxing the locally compact assumption and strengthening to assuming the K_i's are compact, we are slightly weakening the hypotheses in the theorem. In future theorems, we will weaken the hypothesis of locally compact in a similar way. The proof follows:

Proof. First we show that γX is compact. Since γX has the quotient topology induced by h, h is continuous. Also, $h : \alpha X \to \gamma X$ is onto by definition. But since the image of a compact set is compact, and $h(\alpha X) = \gamma X$, we see that γX is compact.

We now show that γX contains X as a dense subspace (that is, $cl_{\gamma X}(X) = \gamma X$). Let $y \in \gamma X$ with $y \in U \in \tau(\gamma X)$. By the definition of quotient topology, $h^{\leftarrow}[U]$ is open in αX. Since h is onto, $h^{\leftarrow}[U]$ is nonempty. Since X is dense in αX, we then have $h^{\leftarrow}[U] \cap X \neq \emptyset$. Let $p \in h^{\leftarrow}[U] \cap X$. Then $h(p) = p$ which implies that $p \in U \cap X$ so that $U \cap X \neq \emptyset$. Therefore γX is contained in $cl_{\gamma X}(X)$ so that X is dense in γX.

We now show that γX is Hausdorff. Define $G = \alpha X\backslash \bigcup_{i=1}^{N} K_i$ and $Q = \{q_i : i = 1, 2, ..., N\}$. Since $\gamma X = G \cup Q$, we must consider three possibilities for distinct points $p, q \in \gamma X$ to show that

γX is Hausdorff. We consider them each in turn:

Case 1: $p, q \in Q$. Set $p = q_n$ and $q = q_m$ with $n \neq m$. Since αX is Hausdorff, Theorem 2.2.11 implies there exists disjoint open subsets U_n, and U_m such that $K_n \subseteq U_n$ and $K_m \subseteq U_m$. Take $V_n = U_n \cap (\alpha X \backslash (K_1 \cup K_2 \cup ... \cup K_{n-1} \cup K_{n+1} ... \cup K_N))$ and $V_m = U_m \cap (\alpha X \backslash (K_1 \cup K_2 \cup ... \cup K_{m-1} \cup K_{m+1} ... \cup K_N))$. Then V_n and V_m are open sets of αX, $V_n \cap K_i = \emptyset$ if $i \neq n$, and $V_m \cap K_i = \emptyset$ if $i \neq m$. Define $V_n^* = h(V_n)$ and $V_m^* = h(V_m)$. Then $V_n = h^{\leftarrow}(V_n^*)$ and $V_m = h^{\leftarrow}(V_m^*)$ (by the definition of h) which makes V_n^* and V_m^* open in γX. By the definition of h and since V_n and V_m are disjoint from the appropriate K_i's respectively, $V_n^* \cap V_m^* = \emptyset, p = q_n \in V_n^*$, and $q = q_m \in V_m^*$. Thus the Hausdorff condition is met in case 1.

Case 2: $p \in G$ and $q \in Q$. Set $q = q_n$. Since singleton's are closed, the normality of αX implies there exists disjoint open subsets U and U_n such that $p \in U$, $K_n \subseteq U_n$, and $p \notin K_n$. Define $V = U \cap (\alpha X \backslash (K_1 \cup K_2 \cup ... \cup K_N))$ and define V_n as in case 1, both open sets in αX. Further define $V^* = h(V)$ and $V_n^* = h(V_n)$. By the same argument as in case 1, we conclude that V^* and V_n^* are disjoint open sets containing p and q respectively. Thus the Hausdorff condition is met.

Case 3: Now suppose $p, q \in G$. There exist disjoint open subsets U_p and U_q of αX such that $p \in U_p$ and $q \in U_q$. Define $V_p = U_p \cap (\alpha X \backslash (K_1 \cup K_2 \cup ... \cup K_N))$ and $V_q = U_q \cap ((\alpha X \backslash (K_1 \cup K_2 ... \cup K_N)$. Then V_p and V_q are open sets of αX, $V_p \cap V_q = \emptyset$, and $h(V_p), h(V_q)$ each contain the same set of points as V_p and V_q respectively. Thus $h(V_p)$ and $h(V_q)$ are disjoint open sets in γX containing p and q respectively.

We conclude that γX is Hausdorff.

□

Definition 3.4.4. The unique compactification of X in Lemma 3.4.3 will be denoted by

$\alpha(X; K_1, K_2, ..., K_n)$.

Corollary 3.4.5. *Let X be a space and let $\{K_i : i = 1, 2, ..., N\}$ be a finite family of mutually disjoint nonempty compact subsets of $\beta X \backslash X$. Then there exists a unique Hausdorff compactification γX of X such that the $\beta - family$ $\mathscr{F}(\gamma X)$ consists of all the sets K_i together with all singletons $\{p\}$,* *where* $p \in [\beta X \backslash X] \backslash \bigcup_{i=1}^{N} K_i$.

Proof. Set $\alpha = \beta$ where the α comes from the statement of Lemma 3.4.3, observe that h is the $\beta - function$ of αX, and apply Lemma 3.4.3.

□

Remark: Using the notation in Definition 3.4.4, we will denote the unique compactification in Corollary 3.4.5 by $\beta(X;K_1,K_2,...,K_n)$.

Lemma 3.4.6. *Let* X *be a space and* αX *be a Hausdorff compactification of* X *with* $\beta - family$ $\mathscr{F}(\alpha X)$. *Suppose that* K_1 *and* K_2 *belong to* $\mathscr{F}(\alpha X)$, K_1 *and* K_2 *are compact, and let* $\mathscr{F}^*$ *consist of the sets of* $\mathscr{F}(\alpha X) \backslash \{K_1, K_2\}$ *together with* $K_1 \cup K_2$. *Then there exists a unique compactification* γX *of* X *whose* $\beta - family$ *is* $\mathscr{F}^*$.

Proof. Let f_α be the $\beta - function$ of αX and $r_1, r_2 \in \alpha X \backslash X$ such that $K_1 = f_\alpha^{\leftarrow}(r_1)$ and $K_2 = f_\alpha^{\leftarrow}(r_2)$. Applying Lemma 3.4.3, Set $\gamma X = \alpha(X;\{r_1,r_2\})$ where $h : \alpha X \to \gamma X$ is a function defined by $h(p) = p$ for $p \in \alpha X \backslash \{r_1, r_2\}$ and $h(r_i) = q, i = 1,2$ where $q \notin \alpha X$. We now show that $\beta - family$ of γX is $\mathscr{F}^*$. Define f_α to be the β function of αX. Consider $h \circ f_\alpha : \beta X \to \gamma X$. Then $h \circ f_\alpha$ is a composition of onto, continuous functions and $h \circ f_\alpha(x) = x$. Thus $h \circ f_\alpha$ is onto and continuous and fixes X. Since $h \circ f_\alpha$ leaves a dense set fixed, Theorem 2.2.4 (and that $h \circ f_\alpha$ is onto and continuous) implies that $h \circ f_\alpha$ is the unique $\beta - function$ for γX. But $(h \circ f_\alpha)^{\leftarrow}(q) = K_1 \cup K_2$ and $(h \circ f_\alpha)^{\leftarrow}(x) = f_\alpha^{\leftarrow}(x)$ for all $x \in \gamma X \backslash \{q\}$. Thus the $\beta - family$ of γX is the $\mathscr{F}$ mentioned in the statement of the lemma. We conclude that γX is the unique compactification of X with the properties specified in the lemma. This follows by the observations after the proof of lemma 3.4.1.

□

Lemma 3.4.7. *Let* X *be a space, and let* K_1 *and* K_2 *be two nonempty compact subsets of* $\beta X \backslash X$. *Then*

$$\beta(X;K_1) \wedge \beta(X;K_2) = \begin{cases} \beta(X;K_1,K_2) & \text{if } K_1 \cap K_2 = \emptyset \\ \beta(X;K_1 \cup K_2) & \text{if } K_1 \cap K_2 \neq \emptyset \end{cases}$$

Proof. Case 1: Suppose $K_1 \cap K_2 = \emptyset$. We begin by comparing $\beta(X;K_1)$ to $\beta(X;K_1,K_2)$. By Lemma 3.4.5, the $\beta - family$ of $\mathscr{F}(\beta(X;K_1) = \{K_1\} \cup \{\{p\} : p \in [\beta X \backslash X] \backslash K_1\}$ and the $\beta - family$ of $\mathscr{F}(\beta(X;K_1,K_2) = \{K_1\} \cup \{K_2\} \cup \{p : p \in [\beta X \backslash X] \backslash K_1\}$. Thus every member of $\mathscr{F}(\beta(X;K_1)$ is contained in a set in $\mathscr{F}(\beta(X;K_1,K_2)$, implying (by Lemma 3.4.1) that $\beta(X;K_1,K_2) \leq \beta(X;K_1)$. Likewise, $\beta(X;K_1,K_2) \leq \beta(X;K_2)$. Therefore, $\beta(X;K_1,K_2)$ is a lower bound for $\beta(X;K_1)$ and $\beta(X;K_2)$. To show $\beta(X;K_1,K_2)$ is the greatest lower bound, begin by assuming γX is a compactification of X such that $\gamma X \leq \beta(X;K_1)$ and $\gamma X \leq \beta(X;K_2)$. Then every member of the $\beta - family$ $\mathscr{F}(\beta(X;K_1))$ is contained in a member of $\mathscr{F}(\gamma X)$ and likewise, every member of the $\beta - family$ $\mathscr{F}(\beta(X;K_2))$ is contained in a member of $\mathscr{F}(\gamma X)$. Thus $K_1 \subseteq A \in \mathscr{F}(\gamma X)$ and $K_2 \subseteq B \in \mathscr{F}(\gamma X)$ for appropriate A,B. Also, $\{p\} \in \mathscr{F}(\gamma X)$ for $p \in [\beta X \backslash X] \backslash K_1 \cup K_2$ since every member of $\mathscr{F}(\alpha(X;K_1)$ is contained in a member of $\mathscr{F}(\gamma X)$ and $\{\{p\} : p \in [\beta X \backslash X] \backslash K_1 \cup K_2\} \subseteq \mathscr{F}(\beta(X;K_2)$. Therefore, $\gamma X \leq \beta(X;K_1,K_2)$, giving the desired conclusion.

Case 2: Now suppose $K_1 \cap K_2 = \emptyset$. An argument just as in case 1 establishes that $\beta(X;K_1 \cup K_2)$ is a lower bound for both $\beta(X;K_1)$ and $\beta(X;K_2)$ (Note that here, Lemma 3.4.5 is applied with $N = 1$ and a single K_i where $K_i = K_1$). Now suppose γX is a lower bound for both $\beta(X;K_1)$ and $\beta(X;K_2)$. As in case 1, $K_1 \subseteq A \in \mathscr{F}(\gamma X)$ and $K_2 \subseteq B \in \mathscr{F}(\gamma X)$ for appropriate A,B and $K_1 \cap K_2 \neq \emptyset$ implies $A \cap B \neq \emptyset$. Since members of $\mathscr{F}(\gamma X)$ partition $\beta X \backslash X, A = B$ implies $K_1 \cup K_2 \subseteq B \in \mathscr{F}(\gamma X)$. Since appropriate singleton sets in $\beta(X;K_1 \cup K_2)$ are contained in γX, we conclude that $\gamma X \leq \beta(X;K_1 \cup K_2)$ (by Lemma 3.4.1)

□

We will use Lemma 3.4.1 freely from this point forward.

Lemma 3.4.8. *Let X a space, and let αX be a Hausdorff compactification of X. Suppose that K_1 and K_2 are sets in $\mathscr{F}(\alpha X)$, and that H_1 and H_2 are nonempty compact subsets of K_1 and K_2 respectively. Then the $\beta - family$ of $\alpha X \wedge \beta(X;H_1 \cup H_2)$ consists of the sets in $\mathscr{F}(\alpha X) \backslash \{K_1,K_2\}$ together with the set $K_1 \cup K_2$.*

Proof. In this proof, we will denote the γX from Lemma 3.4.6 by $\alpha_1(X;K_1 \cup K_2)$. Note that a quick inspection of the proof of Lemma 3.4.6 indicates that $\mathscr{F}(\alpha_1(X;K_1\cup K_2)) = \mathscr{F}(\alpha X)\backslash\{K_1,K_2\}\cup\{K_1\cup K_2\}$.

We first claim that $\alpha X\wedge\beta(X;K_1\cup K_2) = \alpha_1(X;K_1\cup K_2)$. To prove this, first observe that each set in $\mathscr{F}(\alpha X)$ is contained in a set in $\mathscr{F}(\alpha_1(X;K_1\cup K_2))$. Therefore $\alpha X\geq\alpha_1(X;K_1\cup K_2)$. Also, since every element of $\beta(X;K_1\cup K_2)$ is either a singleton set or the set $K_1\cup K_2$, and since $\beta-families$ partition $\beta X\backslash X$, $\beta(X;K_1\cup K_2)\geq\alpha_1(X;K_1\cup K_2)$. Thus $\alpha X\wedge\beta(X;K_1\cup K_2)\geq\alpha_1(X;K_1\cup K_2)$. Next, we must show that if γX is any compactification satisfying $\gamma X\leq\alpha X$ and $\gamma X\leq\beta(X;K_1\cup K_2)$, then $\gamma X\leq\alpha_1(X;K_1\cup K_2)$. Let $K\in\alpha_(X;K_1\cup K_2)$. If $K\in\mathscr{F}(\alpha X)\backslash\{K_1,K_2\}$ then there is a set $L\in\mathscr{F}(\gamma X)$ such that $K\subseteq L$. Also, $K_1\cup K_2\in\beta(X;K_1\cup K_2)$ implies there is an $L_1\in\mathscr{F}(\gamma X)$ such that $K_1\cup K_2\subseteq L_1$. Thus $\gamma X\leq\alpha_1(X;K_1\cup K_2)$. Taken together, we have $\alpha X\wedge\beta(X;K_1\cup K_2) = \alpha_1(X;K_1\cup K_2)$

We next claim that $\alpha X\wedge\beta(X;K_1\cup K_2) = \alpha X\wedge\beta(X;H_1\cup H_2)$. To see this, first observe that since $H_1\cup H_2\subseteq K_1\cup K_2$, $\beta(X;H_1\cup H_2)\geq\beta(X;K_1\cup K_2)$. Therefore, $\alpha X\wedge\beta(X;H_1\cup H_2)\geq\alpha X\wedge\beta(X;K_1\cup K_2) = \alpha_1(X;K_1\cup K_2)$ (where the $\geq$ is by the property that the greatest lower bound preserves order and the equality comes from the claim above). Next, we assume $\alpha X\geq\gamma X$ and $\beta(X;H_1\cup H_2)\geq\gamma X$. We need to show that $\gamma X\leq\alpha_1(X;K_1\cup K_2) = \alpha X\wedge\beta(X;K_1\cup K_2)$. Towards this end, let $K\in\mathscr{F}(\alpha X)$. Then there is a set L in $\mathscr{F}(\gamma X)$ such that $K\subseteq L$. Since $K_1,K_2\in\mathscr{F}(\alpha X)$, there are sets $L_1,L_2\in\mathscr{F}(\gamma X)$ such that $K_1\subseteq L_1$ and $K_2\subseteq L_2$. Also, $H_1\cup H_2\in\mathscr{F}(\beta(X;H_1\cup H_2))$ implies there is an $L_3\in\mathscr{F}(\gamma X)$ such that $H_1\cup H_2\subseteq L_3$. But $H_1\subseteq K_1$ implies that $L_1\cap L_3\neq\emptyset$ so that $L_1 = L_3$ (since $\beta-families$ partition a space). Likewise, $L_2 = L_3$. Thus $H_1\cup H_2\subseteq K_1\cup K_2\subseteq L_1\cup L_2 = L_1$ which implies that $\beta(X;K_1\cup K_2)\geq\gamma X$. Since $\beta(X;K_1\cup K_2)\geq\gamma X$ and $\alpha X\geq\gamma X$, $\alpha X\wedge\beta(X;K_1\cup K_2) = \alpha_1(X;K_1\cup K_2)\geq\gamma X$. Taken together with the above argument, we have $\alpha X\wedge\beta(X;K_1\cup K_2) = \alpha X\wedge\beta(X;H_1\cup H_2)$.

By the two claims, we conclude that

$$\begin{aligned}\mathscr{F}(\alpha X)\backslash\{K_1,K_2\}\cup\{K_1\cup K_2\} &= \mathscr{F}(\alpha_1(X;K_1\cup K_2))\\ &= \mathscr{F}(\alpha X\wedge\beta(X;K_1\cup K_2))\\ &= \mathscr{F}(\alpha X\wedge\beta(X;H_1\cup H_2)).\end{aligned}$$

□

3.5 Dual Points

Definition 3.5.1. Suppose X is a space. We shall refer to a compactification αX of X as a **dual point** of the lattice $\mathscr{K}(X)$ if $\alpha X \neq \beta X$ and there exists no compactification γX different from both αX and βX satisfying $\alpha X < \gamma X < \beta X$.

Lemma 3.5.2. *Suppose X is a space. The space αX is a dual point of $\mathscr{K}(X)$ if and only if there exist distinct elements p and q of $\beta X\backslash X$ such that $\alpha X = \beta(X;\{p,q\})$.*

Proof. ($\Leftarrow$) Suppose $\alpha X = \beta(X;\{p,q\})$. Since $\alpha X = \beta X \Leftrightarrow$ every element of $\mathscr{F}(\alpha X)$ is a singleton, $\alpha X \neq \beta X$. Now suppose there exists a compactification γX such that $\beta(X;\{p,q\}) < \gamma X < \beta X$. Then the $\beta - family$ of γX would include a set strictly smaller than one of the sets in $\beta(X;\{p,q\})$. But this forces all the sets in γX to be singletons, making $\gamma X = \beta X$, a contradiction. Therefore αX is a dual point of $\mathscr{K}(X)$.

($\Rightarrow$) Now assume αX is a dual point of $\mathscr{K}(X)$. Since $\alpha X \neq \beta X$, there exists a set $B \in \mathscr{F}(\alpha X)$ with $|B| \geq 2$. Suppose $|B| \geq 3$. Then there exist distinct $p,q,r \in B$. By the first part of this proof, $\gamma X := \beta(X;\{p,q\})$ is a dual point. But then $\alpha X < \gamma X < \beta X$, contradicting the definition of dual point. Therefore $|B| \leq 2$ for all $B \in \mathscr{F}(\alpha X)$.

Suppose now there exists $B_1, B_2 \in \mathscr{F}(\alpha X)$ such that $|B_1| = |B_2| = 2$. Suppose $x \in B_1$ and $y \in B_2$ such that $x \neq y$. Then $\alpha X < \alpha(X;\{x,y\}) < \beta X$, a contradiction. We conclude that there exists exactly one $B \in \mathscr{F}(\alpha X)$ such that $|B| = 2$, completing the proof.

$\square$

Lemma 3.5.3. *Suppose X is a space and let αX be a Hausdorff compactification of X. Then $\mathscr{F}(\alpha X)$ has either all singleton sets or 2 or more non-singletons if and only if $\alpha X = \beta X$ or there exists distinct compactifications $\gamma_1 X$ and $\gamma_2 X$ such that*

1. *both $\gamma_1 X$ and $\gamma_2 X$ are dual points,*

2. *$\alpha X \wedge \gamma_1 X = \alpha X \wedge \gamma_2 X \neq \alpha X$, and*

3. *the only dual points greater than $\gamma_1 X \wedge \gamma_2 X$ are $\gamma_1 X$ and $\gamma_2 X$.*

Proof. Observe first that $\alpha X = \beta X \Leftrightarrow \mathscr{F}(\alpha X)$ consists entirely of singletons. This completes a portion of the proof in both directions.

($\Rightarrow$) Now assume that $\mathscr{F}(\alpha X)$ has two or more non-singletons. Write $\mathscr{F}(\alpha X) = \{A\} \cup \{B\} \cup \{A_\alpha\}_{\alpha \in I}$ where $|A|, |B| \geq 2$. Suppose $a, b \in A$ and $c, d \in B$ with $a \neq b$ and $c \neq d$. Since members of a $\beta - family$ are distinct, $A \cap B = \emptyset$.

Set $\gamma_1 X = \beta(X; \{a, c\})$ and $\gamma_2 X = \beta(X; \{b, d\})$, clearly distinct compactifications. By Lemma 3.5.2, $\gamma_1 X$ and $\gamma_2 X$ are dual points of $\mathscr{K}(X)$. By Lemma 3.4.8, the $\beta - family$ of $\alpha X \wedge \gamma_1 X$ consists of the sets in $\mathscr{F}(\alpha X) \backslash \{A, B\}$ and $A \cup B$. Applying Lemma 3.4.8 a second time, we observe that the $\beta - family$ of $\alpha X \wedge \gamma_2 X$ also consists of the sets in $\mathscr{F}(\alpha X) \backslash \{A, B\}$ and $A \cup B$. Since compactifications are uniquely determined by their beta families, $\alpha X \wedge \gamma_1 X = \alpha X \wedge \gamma_2 X$, neither of which is αX. Observe that $\gamma_1 X \wedge \gamma_2 X = \beta(X; \{a, c\}, \{b, d\})$ by Lemma 3.4.7. Suppose $\beta(X; \{x, y\})$ is a dual point above $\gamma_1 X \wedge \gamma_2 X$. Then $\{x, y\} \subseteq \{a, c\}$ or $\{x, y\} \subseteq \{b, d\}$. Suppose $\{x, y\} \subseteq \{a, c\}$. Then $\{x, y\} = \{a, c\}$ which implies $\beta(X; \{x, y\}) = \beta(X; \{a, c\})$. If $\{x, y\} \subseteq \{b, d\}$ we conclude that $\beta(X; \{x, y\}) = \beta(X; \{b, d\})$. Therefore, there are no dual points above $\gamma_1 X$ or $\gamma_2 X$. This means $\gamma_1 X$ and $\gamma_2 X$ satisfy requirements (1), (2), and (3), and this direction of the proof is complete.

($\Leftarrow$) Next suppose there exist compactifications $\gamma_1 X$ and $\gamma_2 X$ satisfying (1), (2), and (3). By Lemma 3.5.2 and (1) there exist points $a, b, c, d \in \beta X \backslash X$ such that $\gamma_1 X = \beta(X; \{a, b\})$ and $\gamma_2 X = \beta(X; \{c, d\})$. Observe that $\mathscr{F}(\alpha X \wedge \gamma_1 X) = \mathscr{F}(\alpha X \wedge \gamma_2 X)$ must contain sets $A \supseteq \{a, b\}$ and

$B \supseteq \{c,d\}$, and every set in $\mathscr{F}(\alpha X)$ is contained in some set in $\mathscr{F}(\alpha X \wedge \gamma_1 X) = \mathscr{F}(\alpha X \wedge \gamma_2 X)$. It is clear that $\gamma_1 X \wedge \gamma_2 X = \beta(X;\{a,b\},\{c,d\})$. Suppose towards a contradiction that $\mathscr{F}(\alpha X)$ contains exactly one set which is not a singleton. Call this set K. By Lemma 3.4.5, $\alpha X = \beta(X;K)$. Since $\alpha X \wedge \beta(X;\{a,b\}) \neq \alpha X$, we have that $\alpha X \wedge \beta(X;\{a,b\}) < \alpha X$. If $\{a,b\} \subseteq K$, then $\alpha X \wedge \alpha(X;\{a,b\}) > \alpha X$. Thus it is not the case that both $a,b \in K$. By symmetry, we cannot have both $c,d \in K$. Again using symmetry, this leaves three distinct possibilities for membership in K:

Case 1: $a,b,c,d \notin K$. Since $\alpha X = \beta(X;K)$, $\alpha X \wedge \gamma_1 X = \beta(X;K) \wedge \beta(X;\{a,b\}) = \beta(X;K,\{a,b\})$ where the last equality follows by Lemma 3.4.7. Similarly, $\alpha X \wedge \gamma_2 X = \beta(X;K,\{c,d\})$. By (2), $\alpha X \wedge \gamma_1 X = \alpha X \wedge \gamma_2 X$ so we have $\beta(X;K,\{a,b\}) = \beta(X;K,\{c,d\})$. But this forces $\{a,b\} = \{c,d\}$ which forces $\gamma_1 X = \gamma_2 X$, contradicting that $\gamma_1 X$ and $\gamma_2 X$ are distinct.

Case 2: $a,c \notin K$ and $b,d \in K$. Then $\alpha X \wedge \gamma_1 X = \beta(X;K) \wedge \beta(X;\{a,b\}) = \beta(X;K \cup \{a\})$ where the latter equality follows by Lemma 3.4.7. Similarly, $\alpha X \wedge \gamma_2 X = \beta(X;K) \wedge \beta(X;\{c,d\}) = \beta(X;K \cup \{c\})$. Then since $\alpha X \wedge \gamma_1 X = \alpha X \wedge \gamma_2 X$, $\beta(X;K \cup \{a\}) = \beta(X;K \cup \{c\})$ which implies that $K \cup \{a\} = K \cup \{c\}$. Since $a \notin K$ and $c \notin K$, $a = c$. Since $\gamma_1 X \neq \gamma_2 X$, $b \neq d$. By Lemma 3.4.7, $\beta(X;\{a,b\}) \wedge \beta(X;\{c,d\}) = \beta(X;\{a,b,d\})$. But this implies that $\beta(X;\{a,d\})$ is a dual point of $\mathscr{K}(X)$ greater than $\gamma_1 X \wedge \gamma_2 X$ and different from $\gamma_1 X$ and $\gamma_2 X$, violating (3).

Case 3: $a,b,c \notin K$ but $d \in K$. Using similar reasoning to that used in cases 1 and 2, we conclude that $\alpha X \wedge \beta(X;\{a,b\}) = \beta(X;K,\{a,b\})$ and $\beta X \wedge \beta(X;\{c,d\}) = \beta(X;K \cup \{c\})$. But since $\beta(X;K,\{a,b\}) = \beta(X;K \cup \{c\})$, then $K = K \cup \{c\}$, a contradiction since we assumed $c \notin K$.

This finishes the backwards direction of the proof and thus the proof itself.

$\square$

Lemma 3.5.4. *Let X be locally compact and non-compact, let αX be a Hausdorff compactification of X with $\beta - family$ $\mathscr{F}(\alpha X)$, and let H be a closed subset of $\beta X \backslash X$ containing more than one point. Then $H \in \mathscr{F}(\alpha X)$ if and only if $\beta(X;H) \geq \alpha X$ and there does not exist a compactification of the form $\alpha(X;K)$ such that $\alpha(X;H) > \alpha(X;K) \geq \alpha X$.*

Proof. First assume $H \in \mathscr{F}(\alpha X)$. Since $H \in \mathscr{F}(\alpha X)$, and the remaining elements of $\beta(X;H)$ are singleton sets from $\beta X \backslash X$, $\beta(X;H) \geq \alpha X$. Now suppose towards a contradiction that there exists a compactification of the form $\beta(X;K)$ such that $\alpha(X;H) > \alpha(X;K) \geq \alpha X$. This implies there exists an $L \in \mathscr{F}(\alpha X)$ such that $H \subsetneq K \subseteq L$. If $L = H$, we have $H \subsetneq K \subseteq H$ which implies $H \subsetneq H$, a contradiction. But if $L \neq H$, then we contradict that members of $\mathscr{F}(\alpha X)$ partition $\beta X \backslash X$. The conclusion follows.

Now we assume $H \notin \mathscr{F}(\alpha X)$. If $\beta(X;H) < \beta X$ we are done. Otherwise, $\beta(X;H) \geq \alpha X$. Combining these two facts together, and using that the $\beta - family$ of αX partitions $\beta X \backslash X$, we observe there exists a set $K \in \mathscr{F}(\alpha X)$ such that $H \subsetneq K$. But then we have $\beta(X;H) > \beta(X;K) \geq \alpha X$ as needed.

$\square$

Lemma 3.5.5. *Let X and Y be two spaces such that $\mathscr{K}(X)$ and $\mathscr{K}(Y)$ are lattice isomorphic under the isomorphism Γ. Then the following are true:*

(a) $\Gamma(\beta X) = \beta Y$

(b) δX is a dual point of $\mathscr{K}(X)$ if and only if $\Gamma(\delta X)$ is a dual point of $\mathscr{K}(Y)$.

(c) θX is a dual point of $\mathscr{K}(X)$ above $\alpha X \in \mathscr{K}(X)$ if and only if $\Gamma(\theta X)$ is a dual point above $\Gamma(\alpha X)$.

(d) The same number of dual points lie above δX and $\Gamma(\delta X)$.

Proof. (a) Suppose $\Gamma(\psi X) = \beta Y$. Observe that βX is the unique maximal element of $\mathscr{K}(X)$ and βY is the unique maximal element of $\mathscr{K}(Y)$. Thus $\phi Y \leq \beta Y$ for all compactifications $\phi Y \in \mathscr{K}(Y)$. By Theorem 5.2.10, $\Gamma^{\leftarrow}(\phi Y) \leq \Gamma^{\leftarrow}(\beta Y) = \psi X$. But Γ is onto so as ϕY ranges through all elements in $\mathscr{K}(Y)$, $\Gamma^{\leftarrow}(\phi Y)$ ranges through all elements of $\mathscr{K}(X)$. Therefore ϕX is the unique maximal element of $\mathscr{K}(X)$ which means $\phi X = \beta X$.

(b) Suppose δX is a dual point of $\mathscr{K}(X)$ but $\Gamma(\delta X)$ is NOT a dual point of $\mathscr{K}(Y)$. We know that $\Gamma(\delta X) \neq \beta Y$ or we contradict (a). Now suppose there exists a γY such that $\Gamma(\delta X) < \gamma Y < \beta Y$. Since Γ is onto and by part (a) we have that $\Gamma(\delta X) < \Gamma(\phi X) < \Gamma(\beta Y)$ for some $\phi X \in \mathscr{K}(X)$. Since Γ is order isomorphic, we can apply $\Gamma^{\leftarrow}$ to the inequality to conclude $\delta X < \theta X < \beta X$ which contradicts that δX is a dual point of $\mathscr{K}(X)$. Therefore $\Gamma(\delta X)$ is a dual point of $\mathscr{Y}$. Because Γ is an isomorphism, an identical argument interchanging Γ with $\Gamma^{\leftarrow}$ and X with Y shows the converse.

(c) Using (b), if θX is a dual point above δX, then $\Gamma(\theta X)$ is a dual point of $\mathscr{K}(Y)$. However, $\theta X \geq \delta X$ implies $\Gamma(\theta X) \geq \Gamma(\delta X)$ by Theorem 5.2.10. The conclusion for the forward direction follows. The backwards direction uses a symmetric argument.

(d) This is an immediate consequence of (c).

$\square$

Chapter 4

Main Results

4.1 Lattice Isomorphism and Homeomorphism

We have now arrived at the first main result of this paper.

Theorem 4.1.1. *Suppose X and Y are locally compact, and that Γ is a lattice isomorphism from $\mathscr{K}(X)$ onto $\mathscr{K}(Y)$. Then there exists a homeomorphism h from $\beta X \backslash X$ onto $\beta Y \backslash Y$ such that if $\Gamma(\alpha X) = \alpha Y$ then $\mathscr{F}(\alpha Y) = \{h[H] : H \in \mathscr{F}(\alpha X)\}$. If $\beta X \backslash X$ consists of two elements, then there are two such homeomorphisms. If $|\beta X \backslash X| \neq 2$, there is only one such homeomorphism.*

Before providing the proof, it is useful to summarize its structure. The proof has several parts and proceeds as follows. Here is a description of each part:

Part 1: Prove that the theorem is true for the cases $\beta X \backslash X = \emptyset$, $|\beta X \backslash X| = 1$, and $|\beta X \backslash X| = 2$.

Part 2: Begin the proof that $|\beta X \backslash X| \geq 3$. To do this, we will start by constructing the homeomorphism h.

Part 3: Show the the function h is well-defined.

Part 4: Show that the function h has a special property with respect to closed sets. This will be used later in the proof.

Part 5: Define a function k which is to be the inverse of h. We briefly show that k has the same properties as h. We also show that k has a special property similar to the property of h shown in part 4.

Part 6: Show that h and k are inverses of one another, making h bijective.

Part 7: Show that h and k are continuous.

Part 8: Show that h satisfies the property that $\mathscr{F}(\alpha Y) = \{h[H] : H \in \mathscr{F}(\alpha X)\}$.

Part 9: Conclude by showing that h is a unique function satisfying the conclusion of the theorem.

Proof. Part 1:

First we note that if $\beta X \backslash X = \emptyset$ then $\beta X = X$, $\beta Y = Y$, and the empty function has the required properties (albeit trivially).

Now suppose that $\beta X \backslash X$ consists of one element, say $\beta X \backslash X = \{p\}$. Then there is only one beta family, $\{\{p\}\}$ for all compactifications. Thus $|\mathscr{K}(X)| = 1$. Since $\mathscr{K}(X)$ is lattice isomorphic to $\mathscr{K}(Y)$, this implies $|\mathscr{K}(Y)| = 1$ as well. Thus $|\beta Y \backslash Y| = 1$. Otherwise, $a, b \in \beta Y \backslash Y, a \neq b$ would imply that βY and $\beta(Y; \{a, b\})$ were distinct compactifications of Y. Setting $\beta Y \backslash Y = \{q\}$, it is clear that the function $h : \beta X \backslash X \to \beta Y \backslash Y$ defined by $h(p) = q$ is a homeomorphism satisfying the required properties.

Now suppose that $|\beta X \backslash X| = 2$. Set $\beta X \backslash X = \{a, b\}$. There are only two partitions of $\{a, b\}$, and they are $\{\{a\}, \{b\}\}$ and $\{\{a, b\}\}$. Since the $\beta - function$ for βX is the identity on βX, the $\beta - family$ for βX is $\{\{a\}, \{b\}\}$. Also, that means $\{\{a, b\}\}$ is the $\beta - family$ for a one-point compactification of X by the definition of one-point compactification and by Corrollary 3.4.5. Therefore $|\mathscr{K}(X)| = 2$. Since $\mathscr{K}(X)$ and $\mathscr{K}(Y)$ are lattice isomorphic, this implies that $|\mathscr{K}(Y)| = 2$ as well. Thus $\beta Y \backslash Y$ has at least two elements. Suppose $\beta Y \backslash Y$ has three or more elements. Then $\mathscr{K}(Y)$ would have at least three dual points while $\mathscr{K}(X)$ has only one dual point. This violates Lemma 3.5.5(d). Therefore $|\beta Y \backslash Y| = \{c, d\}$ and $\mathscr{K}(Y) = \{\beta Y, \beta(Y; \{c, d\})\}$ for some $c, d \in \beta Y \backslash Y$.

Set $\beta Y \backslash Y = \{c, d\}$. Now define functions h_1, h_2 from $\beta X \backslash X$ to $\beta Y \backslash Y$ where h_1 maps a to c and b to d while h_2 maps a to d and b to c. However, since βX is Hausdorff, Lemma 2.2.14

implies that $\beta X \backslash X$ is Hausdorff. Since every subset in a two-point Hausdorff space is open, this means that h_1 is continuous. Similar reasoning implies that h_2 is continuous. Since h_1 and h_2 are bijective, we conclude they are homeomorphisms. Now suppose that $\Gamma(\alpha X) = \alpha Y$. If $\alpha = \beta$, then $\Gamma(\beta X) = \beta Y$ where $\beta X = \beta(X; \{\{a\}, \{b\}\}$ and $\beta Y = \beta(Y; \{\{c\}, \{d\}\}$. Then $h_1(a) = c, h_1(b) = d, h_2(a) = d$, and $h_2(b) = c$. Thus $\mathscr{F}(\alpha Y) = \{h[H] : H \in \mathscr{F}(\alpha X)\}$. Similarly, if $\alpha X = \beta(X; \{a,b\}$ then $\Gamma(\beta X; \{a,b\}) = \beta(Y; \{c,d\})$ implies $\mathscr{F}(\alpha Y) = \{h[H] : H \in \mathscr{F}(\alpha X)\}$ as well. Thus the theorem is verified in the case that $|\beta X \backslash X| = 2$.

Part 2:

Now suppose that $\beta X \backslash X$ has three or more elements. We define a mapping $h : \beta X \backslash X \to \beta Y \backslash Y$ in the following way: First let $p \in \beta X \backslash X$ and let q, r be any other elements in $\beta X \backslash X$ such that $|\{p,q,r\}| = 3$. By Lemma 3.5.2, $\beta(X; \{p,q\})$ and $\beta(X; \{p,r\})$ are dual points of $\mathscr{K}(X)$. By Lemma 3.5.5(b), we conclude that $\Gamma(\beta(X; \{p,q\}) = \beta(Y; \{a,b\}))$ and $\Gamma(\beta(X; \{p,r\})) = \beta(Y; \{c,d\})$ for some $a,b,c,d \in \beta Y \backslash Y$. Since $\{p,q\} \cap \{p,r\} \neq \emptyset$, Lemma 3.4.7 implies that $\beta(X; \{p,q\}) \wedge \beta(X; \{p,r\}) = \beta(X; \{p,q,r\})$. Using the isomorphism property, this gives us that

$$\begin{aligned}
\Gamma(\beta(X); \{p,q,r\}) &= \Gamma(\beta(X; \{p,q\}) \wedge \beta(X; \{p,r\})) \\
&= \Gamma(\beta(X; \{p,q\})) \wedge \Gamma(\beta(X; \{p,r\})) \\
&= \beta(Y; \{a,b\}) \wedge \beta(Y; \{c,d\}).
\end{aligned}$$

Suppose towards a contradiction that $\{a,b\} \cap \{c,d\} = \emptyset$. Then Lemma 3.4.7 implies that $\beta(Y; \{a,b\}) \wedge \beta(Y; \{c,d\}) = \beta(Y; \{a,b\}, \{c,d\})$. Observe that $\beta(Y; \{a,b\})$ and $\beta(Y; \{c,d\})$ are the dual points of $\mathscr{K}(Y)$ above $\beta(Y; \{a,b\}, \{c,d\})$ while $\beta(X; \{p,q\})$, $\beta(X; \{p,r\})$, and $\beta(X; \{q,r\})$ are the dual points of $\mathscr{K}(X)$ above $\beta(X; \{p,q,r\})$ (This uses Lemma 3.4.1). By Lemma 3.5.5(c), there are the same number of dual points above both $\beta(Y; \{a,b\}, \{c,d\})$ and $\beta(X; \{p,q,r\})$ in their respective lattices. This contradiction implies that $\{a,b\} \cap \{c,d\} \neq \emptyset$. If the two sets are the same, so are the dual points $\beta(Y; \{a,b\}))$ and $\beta(Y; \{c,d\})$ which contradicts that Γ is one-to-one. Therefore $|\{a,b\} \cap \{c,d\}| = 1$. We assume without loss of generality that $\{a,b\} \cap \{c,d\} = a$. We now define h by $h(p) = a$.

Part 3:

Next, we check that h is well-defined. To see this, we need to show that a does not depend on the choice of q or r. Suppose $s \in \beta X \backslash X$ such that $|\{p,q,r,s\}| = 4$. By the isomorphism property, there exists points $\{y,z\}$ such that $\Gamma(\beta(X;\{p,s\})) = \beta(Y;\{y,z\})$. Without loss of generality, we can assume that $\Gamma(\beta(X;\{p,r\}) = \beta(Y;\{a,c\})$. Recall that $\Gamma(\beta(X;\{p,q\}) = \beta(Y;\{a,b\})$. Applying the same argument as above, we observe that $\{y,z\}$ intersects $\{a,b\}$ and $\{a,c\}$ in exactly one point. Suppose towards a contradiction that $a \notin \{y,z\}$. This forces $\{y,z\} = \{b,c\}$. Applying Lemma 3.4.7, we have that

$$\begin{aligned}\beta(X;\{p,q\}) \wedge \beta(X;\{p,r\}) \wedge \beta(X;\{p,s\}) &= \beta(X;\{p,q,r\}) \wedge \beta(X;\{p,s\}) \\ &= \beta(X;\{p,q,r,s\}).\end{aligned}$$

On the other hand, we have that

$$\begin{aligned}\Gamma(\beta(X;\{p,s\}) \wedge \beta(X;\{p,r\}) \wedge \beta(X;\{p,q\})) &= \Gamma(\beta(X;\{p,s\})) \wedge \Gamma(\beta(X;\{p,r\})) \wedge \Gamma(\beta(X;\{p,q\})) \\ &= \beta(Y;\{y,z\}) \wedge \beta(Y;\{a,c\}) \wedge \beta(Y;\{a,b\}) \\ &= \beta(Y;\{b,c\}) \wedge \beta(Y;\{a,c\}) \wedge \beta(Y;\{a,b\}) \\ &= \beta(Y;\{a,b,c\}) \wedge \beta(Y;\{a,b\}) \\ &= \beta(Y;\{a,b,c,\}).\end{aligned}$$

That is, $\Gamma(\beta(X;\{p,q,r,s\}) = \beta(Y;\{a,b,c\})$. Since $\mathscr{K}(X)$ and $\mathscr{K}(Y)$ are isomorphic, this is impossible (observe that the domain compactification and the codomain compactification have different structure). To be more precise, we observe that there are six dual points greater than

$\beta(X;\{p,q,r,s\})$ (which have the form $\beta(X;\{t_1,t_2\})$ where $\{t_1,t_2\}$ varies between the six possible two point subsets of $\{p,q,r,s\})$ while, just as earlier with $\beta(X;\{p,q,r\})$, there are three dual points greater than $\beta(Y;\{a,b,c\})$. This contradiction arose by assuming that $a \notin \{y,z\}$. Thus $a \in \{y,z\}$. This means that given two points $s,t \in [\beta X\backslash X]\backslash\{p\}$, if $\Gamma(\alpha(X;\{p,s\}) = \alpha(Y;\{y,z\})$, then $a \in \{y,z\}$ and if $\Gamma(\alpha(X;\{p,t\}) = \alpha(Y;\{y_1,z_1\})$ then $a \in \{y_1,z_1\}$, and $\{y,z\} \cap \{y_1,z_1\} = \{a\}$, making the choice of a unique in $h(p) = a$.

We have established that h is a well-defined function from $\beta X\backslash X$ to $\beta Y\backslash Y$. It remains to show that h is a bijective continuous function with a continuos function (a homeomorphism). Towards this end, we do some preliminary work that will later be used to show that h and $h^{\leftarrow}$ are continuous.

Part 4:

Suppose H be a closed subset of $\beta X\backslash X$ with $|H| \geq 2$. By Lemma 3.5.3, $\Gamma(\beta(X;H)) = \beta(Y;K)$ for some closed subset $K \subseteq \beta Y\backslash Y$ with $|K| > 2$ by (Lemma 3.5.5(d)). That is, if $\beta(Y;K)$ did not have this form, Lemma 3.5.3 would imply the existence of distinct dual points $\gamma_1 Y$ and $\gamma_2 Y$ with the properties specified in Lemma 3.5.3, implying the same occurs in $\mathscr{K}(X)$, which is a contradiction (given the form of $\beta(X;H)$). Suppose $p,q \in H$ with $|\{p,q\}| = 2$. Then $\beta(X;\{p,q\}) \geq \beta(X;H)$ which gives us $\beta(Y;\{a,b\}) = \Gamma(\beta(X;\{p,q\})) \geq \Gamma(\beta(X;H)) = \beta(Y;K)$ by Theorem 5.2.10, Lemma 3.5.5(b), and for some points $a,b \in \beta Y\backslash Y$. Since $\alpha(Y;\{a,b\}) \geq \alpha(Y;K)$, we have that $\{a,b\} \subseteq K$. Recall that in the definition of h, if $\Gamma(\beta(X;\{p,q\}) = \beta(Y;\{a,b\})$, then $h(p) \in \{a,b\}$ and $h(q) \in \{a,b\}$. Therefore, $h(\{p,q\}) \subseteq \{a,b\} \subseteq K$. But since p was an arbitrary member of H, we conclude that $h(H) \subseteq K$.

Part 5:

We now define a new function k from $\beta Y\backslash Y$ to $\beta X\backslash X$ which is to be the inverse of h. Since $\mathscr{K}(X)$ is isomorphic to $\mathscr{K}(Y)$, we conclude that since $\beta X\backslash X$ has at least three elements, so does $\beta Y\backslash Y$ (This follows by the statement "$|\{a,b\} \cap \{c,d\}| = 1$" from part 2 of the proof). Now let $a \in \beta Y\backslash Y$ and choose two points $b,c,\in \beta Y\backslash Y$ such that $|\{a,b,c\}| = 3$. Then just as earlier, $\beta(Y;\{a,b\})$ and $\beta(Y;\{a,c\})$ are dual points of $\mathscr{K}(Y)$ while $\Gamma^{\leftarrow}(\beta(Y;\{a,b\}) = \beta(X;\{p,q\})$

and $\Gamma^{\leftarrow}(\beta(Y;\{a,c\}) = \beta(X;\{r,s\})$ for points $p,q,r,s \in \beta X \backslash X$. Also as in the earlier disucssion, $|\{p,q\} \cap \{r,s\}| = 1$, say $\{p,q\} \cap \{r,s\} = \{r\}$ where r does not depend on the choice of points b and c. We define $k(a) = r$. Just as earlier we can show that $k(K) \subseteq H$ where K and H are the same closed subsets as above.

Part 6:

We are now prepared to show that h and k are inverses of one another, making h bijective. To do this, we will show that $k \circ h$ is the identity mapping on $\beta X \backslash X$ and that $h \circ k$ is the identity mapping on $\beta Y \backslash Y$. To show $k \circ h$ is the identity mapping on $\beta X \backslash X$, begin by choosing points $p,q \in \beta X \backslash X$ such that $|\{p,q\}| = 2$. Then $\Gamma(\beta(X;\{p,q\}) = \beta(Y;\{a,b\})$ and by earlier work, $h(p) \in \{a,b\}$. Without loss of generality, we assume that $h(p) = a$. Now suppose towards a contradiction that $k(a) \neq p$. Then by earlier work as well, $k(a) = q$. Now choose a point $r \in \beta X \backslash X$ such that $|\{p,q,r\}| = 3$ and consider $\Gamma(\beta(X;\{p,r\})) = \beta(Y;\{a,c\})$, where $\beta(Y;\{a,c\})$ has this form based on earlier work. Then $k(a) \in \{p,r\}$ and since $k(a) \neq p$, we have that $k(a) = r$. But this means $k(a) = q = r$ which is a contradiction since q and r are distinct. Therefore $k(a) = p$ which implies that $k \circ h$ is the identity mapping on $\beta X \backslash X$. A symmetric argument can be used to show that $h \circ k$ is the identity map on $\beta Y \backslash Y$. Therefore h is a bijection between $\beta X\ X$ and $\beta Y \backslash Y$ with $k = h^{\leftarrow}$.

Part 7:

Now, suppose that H is a closed subset of $\beta X \backslash X$ containing at least two points so that $\Gamma(\beta(X;H) = \beta(Y;K)$ for some closed subset K of $\beta Y \backslash Y$. As noted earlier, $h(H) \subseteq K$. Also, $k(K) \subseteq H$ based on earlier work so applying h to both sides we have $h \circ k(K) \subseteq h(H)$ and $K \subseteq h(H)$. Taken together, $h(H) = K$. This means that h maps closed sets to closed sets which implies that $h^{\leftarrow}$ is continuous. A symmetric argument shows that k maps closed sets to closed sets so that $k^{\leftarrow}$ is continuous. Therefore, h is a homeomorphism between $\beta X \backslash X$ and $\beta Y \backslash Y$.

Part 8:

We now show that the homeomorphism h satisfies the property that if $\Gamma(\alpha X) = \alpha Y$ then $\mathscr{F}(\alpha Y) = \{h[H] : H \in \mathscr{F}(\alpha X)\}$. Suppose αX is a compactification of X such that $\Gamma(\alpha X) = \alpha Y$.

Let $H \in \mathscr{F}(\alpha X)$ and suppose first that $|H| > 1$. Then $\beta(X;H) \geq \alpha X$. From Lemma 3.5.4, we know there does not exist a compactification of the form

$$\beta(X;H) > \beta(X;V) \geq \alpha X \tag{4.1}$$

where $V \in \beta X \backslash X$. Next, observe that $\Gamma(\beta(X;H) = \beta(Y;K)$ where $h(H) = K$, K closed in $\beta Y \backslash Y$ and $\beta(Y;h(H)) \geq \alpha Y$. Suppose towards a contradiction that $\beta(Y;h(H)) > \beta(Y;W) \geq \alpha Y$ for some $W \in \beta X \backslash X$. Applying $\Gamma^{\leftarrow}$to both sides of this inequality and using Theorem 5.2.10 we have

$$\begin{aligned}\Gamma^{\leftarrow}(\beta(Y;h(H)) > (\beta(Y;W)) \geq (\alpha Y)) &= \Gamma^{\leftarrow}\beta(Y;h(H)) > \Gamma^{\leftarrow}(\beta(Y;W)) \geq \Gamma^{\leftarrow}(\alpha Y) \\ &= \beta(X;H) > \beta(X;U) \geq \beta Y\end{aligned}$$

for some $U \in \beta X \backslash X$. But this contradicts (4.1). Therefore, there is no compactification of the form $\beta(Y;W)$ such that $\beta(Y;h(H)) > \beta(Y;W) \geq \alpha Y$. Another application of Lemma 3.5.4 implies that $h(H) \in \mathscr{F}(\alpha Y)$ as needed.

Now let $K \in \mathscr{F}(\alpha Y)$. Then just as above, $\beta(Y;K) \geq \beta Y$. From Lemma 3.5.4, there does not exist a compactification of the form

$$\beta(Y;K) > \beta(Y;W) \geq \alpha Y \tag{4.2}$$

where $W \in \beta Y \backslash Y$. Next, observe that $\Gamma^{\leftarrow}(\beta(Y;K) = \beta(X;H)$ where $k(K) = H$, H closed in $\beta X \backslash X$ and $\beta(X;k(K)) \geq \alpha X$. Suppose towards a contradiction that $\beta(X;k(K)) > \beta(X;V) \geq \alpha X$ for some $V \in \beta Y \backslash Y$. Applying Γ to both sides of this inequality and using Theorem 5.2.10 we have

$$\Gamma(\beta(X;H)) > (\beta(X;V)) \geq (\alpha X)) = \Gamma(\beta(X;H) > \Gamma(\beta(X;V)) \geq \Gamma(\alpha X)$$
$$= \beta(Y;K) > \beta(Y;T) \geq \beta Y$$

for some $T \in \beta X \backslash X$. But this contradicts 4.2. Therefore, there is no compactification of the form $\beta(X;V)$ such that $\beta(X;H) > \beta(X;V) \geq \alpha X$. Applying Lemma 3.5.4 gives that $H \in \mathscr{F}(\alpha X)$. But therefore K has the form $K = h \circ k(K) = h(H)$ which implies that $K \in \{h[H] : H \in (X)\}$.

We have so far shown that if $\Gamma(\alpha X) = \alpha Y$, then $\mathscr{F}(\alpha Y) = \{h[H] : H \in \mathscr{F}(\alpha X)\}$ when singleton sets are not considered. Now suppose that $\{p\} \in \mathscr{F}(\alpha X)$. We claim that $\{h(p)\} \in \mathscr{F}(\alpha Y)$. Suppose towards a contradiction that $\{h(p)\} \notin \mathscr{F}(\alpha Y)$. Then there exists a set $K \in \mathscr{F}(\alpha Y)$ such that $h(p) \in K$ and $|K| > 1$. This implies that $p \in h^{\leftarrow}(K)$. But from earlier, we know that $h^{\leftarrow}(K) \in \mathscr{F}(\alpha X)$. This means p is contained in a member of the $\beta - family$ of αX with cardinarily at least 2. But this contradicts that $\{p\} \in \mathscr{F}(\alpha X)$ since the $\beta - family$ of αX partitions $\beta X \backslash X$.

Now suppose $\{q\} \in \mathscr{F}(\alpha Y)$. We claim that $\{h^{\leftarrow}(q)\} \in \mathscr{F}(\alpha X)$. Suppose $\{h^{\leftarrow}(q)\} \notin \mathscr{F}(\alpha X)$. Then there exists a set $H \in \mathscr{F}(\alpha X)$ containing more than one point such that $h^{\leftarrow}(q) \in H$. This means that $q \in h(H)$ where $|h(H)| > 1$. But from earlier, we know that $h(H) \in \mathscr{F}(\alpha Y)$. This means q is contained in a member of the $\beta - family$ of αY with cardinality at least 2. But this contradicts that $\{q\} \in \mathscr{F}(\alpha Y)$ since the $\beta - family$ of αY partitions $\beta Y \backslash Y$. It remains to show that h is the only homeomorphism satisfying the property that if $\Gamma(\alpha X) = \alpha Y$ then $\mathscr{F}(\alpha Y) = \{h[H] : H \in \mathscr{F}(\alpha X)$.

Part 9:

Suppose that l is a homeomorphism mapping $\beta X \backslash X$ to $\beta Y \backslash Y$ such that for any compactification αX of X, if $\Gamma(\alpha X) = \alpha Y$ then $\mathscr{F}(\alpha Y) = \{l[H] : H \in \mathscr{F}(\alpha X)\}$. Just like when we began in defining h, let p be a point in $\beta X \backslash X$ and choose two other points q, r in $\beta X \backslash X$ such that

$|\{p,q,r\}| = 3$. As earlier, there exist points $a,b,c \in \beta Y \backslash Y$ such that $\Gamma(\alpha(X;\{p,q\})) = \alpha(Y;\{a,b\})$ and $\Gamma(\alpha(X;\{p,r\})) = \alpha(Y;\{a,c\})$ where we define $h(p) = a$. Since Γ is an isomorphism, Γ is one-to-one and $b \neq c$. Since $\mathscr{F}(\alpha Y) = \{h[H] : H \in \mathscr{F}(\alpha X)\}$, $l(\{p,q\}) \subseteq \{a,b\})$. Suppose towards a contradiction that $l(p) = b$. But $l(\{p,r\}) \subseteq \{a,c\}$ which implies that $l(p) = a$ or $l(p) = c$. Since a,b, and c are distinct, this contradictions that $l(p) = b$. Thus $l(p) = a$ and $l = h$.

The proof is complete.

□

As a quick observation, we observe that the previous theorem implies that if $\mathscr{K}(X)$ is lattice isomorphic to $\mathscr{K}(Y)$ then $|\beta X \backslash X| = |\beta Y \backslash Y|$.

The following is a converse of the previous theorem.

Theorem 4.1.2. *Suppose that X and Y are locally compact and that h is a homeomorphism from $\beta X \backslash X$ to $\beta Y \backslash Y$. Let αX be any compactification of X with $\beta - family$ $\mathscr{F}(\alpha X)$. Then there exists a unique compactification αY of Y whose $\beta - family$ is $\{h(H) : H \in \mathscr{F}(\alpha X)\}$, and the mapping Γ defined by $\Gamma(\alpha X) = \alpha Y$ is a lattice ismorphism from $\mathscr{K}(X)$ to $\mathscr{K}(Y)$.*

Proof. If X is compact then $\beta X \backslash X = \emptyset$ and the situation is trivial. Now suppose X is non-compact and define f_α to be the $\beta - function$ from βX to αX. Then the function $f_\alpha \circ h^\leftarrow$ is continuous and onto from $\beta Y \backslash Y$ to $\alpha X \backslash X$ (the composition of continuous functions is continuous and the composition of onto functions is onto). Set $\alpha Y = Y \dot{\bigcup} [\alpha X \backslash X]$, define a function k from βY to αY by

$$k(p) = \begin{cases} (f_\alpha \circ h^\leftarrow)(p) & \text{if } p \in \beta Y \backslash Y \\ p & \text{if } p \in Y \end{cases}.$$

Since f_α and $h^\leftarrow$ are onto, so is k. Equip αY with the quotient topology σ induced by k. Then k is continuous by design. Since $k(\beta Y) = \alpha Y$, and βY is compact, αY is compact. To see that Y is dense in αY, we proceed as in the proof to Theorem 3.4.3. Let $y \in \alpha Y$ with $y \in U \in \tau(\alpha Y)$. Then $k^\leftarrow[U]$ is open in βY and $k^\leftarrow[U]$ is non-empty. Y is dense in βY so $k^\leftarrow[U] \cap Y \neq \emptyset$. Let

$q \in K^{\leftarrow}[U] \cap Y$. Then $h(q) = q$ which implies $p \in U \cap Y$ so $U \cap Y \neq \emptyset$. Thus $\alpha Y \subseteq cl_{\alpha Y}(Y)$, $cl_{\alpha Y}(Y) = \alpha Y$, and Y is dense in αY.

At this point, we know that αY is a compactification of Y. To show that αY is Hausdorff, we first prove two claims regarding the quotient topology σ.

Claim 1: $\tau(Y) = \sigma(Y)$: Let $U \in \tau(Y)$. Then $U = U_1 \cap Y$ for some $U_1 \in \tau(\beta Y)$. We have $k^{\leftarrow}[U] = U$. However, since Y is a locally compact, dense subset of a compact Hausdorff space, Corollary 8.14 in [6] implies that Y is open in βY. Thus U is open in $\tau(\beta Y)$, making U open in $\tau(\sigma)$. Now let $U \in \sigma(Y)$. Then $k^{\leftarrow}[U] = U \in \tau(\beta Y)$. Thus, $U \in \tau(Y)$.

Claim 2: $\tau(\alpha X \backslash X) = \sigma(\alpha Y \backslash Y)$: First let $A \subseteq \alpha X \backslash X$ be closed in $\tau(\alpha X \backslash X)$. Since X is open in $\beta X \backslash X$ (see the proof of claim 1 above), $\beta X \backslash X$ is closed in βX (and thus compact by 2.2.9). Thus $f_{\alpha}^{\leftarrow}[A]$ is closed in $\beta X \backslash X$. Since h is a homeomorphism, we have that $h \circ f_{\alpha}^{\leftarrow}[A] = (f_{\alpha} \circ h^{\leftarrow})^{\leftarrow}[A] = k^{\leftarrow}[A]$ is closed as well (but this time in $\beta Y \backslash Y$. Thus A is closed in $\sigma(\alpha Y \backslash Y)$. Since every closed set in $\tau(\alpha X \backslash X)$ is contained in every closed set of $\sigma(\alpha Y \backslash Y)$, using set complements we have $\tau(\alpha X \backslash X) \subseteq \sigma(\alpha Y \backslash Y)$. Since Hausdorff property is expansive, we observe that $\sigma(\alpha Y \backslash Y)$ is Hausdorff.

Now suppose that A is closed in $\sigma(\alpha Y \backslash Y)$. Since k restricted to $\beta Y \backslash Y$ is continuous, $k^{\leftarrow}[A]$ is closed in $\beta Y \backslash Y$. Since $\beta Y \backslash Y$ is compact and Hausdorff (This can be shown to be compact just as $\beta X \backslash X$ was shown to be compact), $k^{\leftarrow}[A]$ is also compact (2.2.9). Observe that $k^{\leftarrow}[A] = h[f_{\alpha}^{\leftarrow}[A]]$ which implies $h^{\leftarrow}[k^{\leftarrow}[A]] = f_{\alpha}^{\leftarrow}[A]$. Since h is a homeomorphism, $h^{\leftarrow}$ maps compact sets to compact sets so $f_{\alpha}^{\leftarrow}[A]$ is compact. But another application of 2.2.9 implies that $f_{\alpha}^{\leftarrow}[A]$ is closed in $\beta X \backslash X$. Since continuous functions map compact sets to compact sets, $A = f_{\alpha} \circ f_{\alpha}^{\leftarrow}[A]$ (Note: equality here requires f_{α} to be onto) is also compact and thus closed. As above, this implies $\sigma(\alpha Y \backslash Y) \subseteq \tau(\alpha X \backslash X)$. We conclude that $\tau(\alpha X \backslash X) = \sigma(\alpha Y \backslash Y)$.

Next we show that $\alpha Y = Y \dot{\cup} [\alpha X \backslash X]$ Is Hausdorff. We consider three cases.

Case 1: $p, q \in Y$. Since $Y \subseteq \beta Y$, βY is Hausdorff, and Hausdorff is hereditary, there exists $U, V \in \tau(Y)$ such that $p \in U, q \in V$ and $U \cap V = \emptyset$. But $k^{\leftarrow}[U] = U$ is open in $\sigma(Y)$ and $k^{\leftarrow}[V] = V$

is open in $\sigma(Y)$ by claim 1.

Case 2: Suppose $p,q \in \alpha Y \backslash Y = \alpha X \backslash X$. By claim 2, we have that $\alpha Y \backslash Y$ is compact and Hausdorff with the quotient topology σ. Therefore $\alpha Y \backslash Y$ is normal. By an equivalent condition to normality (See 5.15(c) in [6]), there exist two open sets G_p and G_q of $\alpha Y \backslash Y$ such that $p \in G_p, q \in G_q$, and $cl_{\alpha Y \backslash Y}(G_p) \cap cl_{\alpha Y \backslash Y}(G_q) = \emptyset$. A basic result regarding subspaces states that if $B \subseteq A \subseteq Z$ where Z is a space, then $cl_A(B) = cl_X(B) \cap A$. Thus $cl_{\alpha Y \backslash Y}(G_p) = cl_{\alpha Y}(G_p) \cap \alpha Y \backslash Y = cl_{\alpha Y}(G_p)$ since $G_p \subseteq \alpha Y \backslash Y$. Likewise for G_q. In particular, $cl(G_p)$ and $cl(G_q)$ are closed in αY (where the closure symbol is unambiguous in light of the prior statement). Since βY is normal, the disjoint closed subsets $k^{\leftarrow}[cl(G_p)]$ and $k^{\leftarrow}[cl(G_q]$ are contained in disjoint open sets H_p and H_q of βY.

Consider the open sets $k^{\leftarrow}[G_p] \cup Y$, and $k^{\leftarrow}[G_q] \cup Y$ in βY. Set $U_p = (k^{\leftarrow}[G_p] \cup Y) \cap H_p$ and $U_q = (k^{\leftarrow}[G_q] \cup Y) \cap H_q$, both open sets of βY. Then we have $U_p = (k^{\leftarrow}[G_p] \cup Y) \cap H_p = (k^{\leftarrow}[G_p] \cap H_p) \cup (Y \cap H_p) = k^{\leftarrow}[G_p] \cup (Y \cap H_p)$ which in turn equals $k^{\leftarrow}[G_p \cup (Y \cap H_p)]$ since $Y \cap H_p \subseteq Y$ and k is fixed on Y. Likewise, $U_q = k^{\leftarrow}[G_q \cup (Y \cap H_q)]$. By the definition of quotient topology, we see that $G_p \cup (Y \cap H_p)$ and $G_q \cup (Y \cap H_q)$ are open. Since $p \in G_p \subseteq U_p$ and $q \in G_q \subseteq U_q$, $p \in U_p$ and $q \in U_q$. Since $U_p \subseteq H_p, U_q \subseteq H_q$, and $H_p \cap H_q = \emptyset$, U_p and U_q are disjoint. Therefore p and q are contained in disjoint open sets of αY.

Case 3: Suppose $p \in Y$ and $q \in \alpha Y \backslash Y$. Since Y is locally compact, Hausdorff, and open, Theorem 8.12(b) of [6] implies there exists an open set U_p of βY such that $p \in U_p \subseteq cl_{\beta Y}(U_p) \subseteq Y$. Since Y is open in βY, U_p is open in Y. By claim 1, $U_p \in \sigma(Y)$. Also, $k^{\leftarrow}[cl_{\beta Y}(U_p)] = cl_{\beta Y}(U_p)$ is closed in αY. Set $V = \alpha Y \backslash cl_{\beta Y}(U_p)$. Then $q \in V$ so we have $p \in U_p$ and $q \in V$ with $U_p \cap V = \emptyset$.

Therefore, αY is Hausdorff.

Next, we show that $\mathscr{F}(\alpha Y) = \{h(H) : H \in \mathscr{F}(\alpha X)\}$. Observe that since k is a continuous function from βY onto αY leaving Y fixed, k is the $\beta - function$ of αY (see Lemma 3.1.1). Therefore, we need only analyze this function to determine the $\beta - family$. Suppose K is in the $\beta - family$ $\mathscr{F}(\alpha Y)$. Then $K = k^{\leftarrow}(q)$ for some $q \in \alpha Y \backslash Y = \alpha X \backslash X$. But $k^{\leftarrow}(q) = (f_\alpha \circ h^{\leftarrow})^{\leftarrow}(q) = h(f_\alpha^{\leftarrow}(q)) = h(H)$ where $H = f_\alpha^{\leftarrow}(q) \in \mathscr{F}(\alpha X)$. Conversely, suppose $H \in \mathscr{F}(\alpha X)$.

Then $H = f_\alpha^{\leftarrow}(q)$ for some $q \in \alpha X \backslash X$ so $h(H) = h(f_\alpha^{\leftarrow}(q)) = (f_\alpha \circ h^{\leftarrow})(q) = k^{\leftarrow}(q) \in \mathscr{F}(\alpha Y)$. Thus $\mathscr{F}(\alpha Y) = \{h(H) : H \in \mathscr{F}(\alpha X)\}$. Since compactifications are uniquely determined by their $\beta - families$, αY is the unique Hausdorff compactification of Y whose $\beta - family$ is $\{h[H] : H \in \mathscr{F}(\alpha X)\}$.

It remains to show that the mapping Γ defined by $\Gamma(\alpha X) = \alpha Y$ is a lattice isomorphism from $\mathscr{K}(X)$ to $\mathscr{K}(Y)$. First we show one-to-one. Let $\alpha_1 X$ and $\alpha_2 X$ be distinct (up to isomorphism) compactifications of X. Then $\Gamma(\alpha_1 X) = \alpha_2 Y$ and $\Gamma(\alpha_2 X) = \alpha_2 Y$ have distinct $\beta - families$ $\{h(H) : H \in \mathscr{F}(\alpha_1 X)\}$ and $\{h(H) : H \in \mathscr{F}(\alpha_2 X)\}$. Thus, $\mathscr{F}(\alpha_1 X)$ and $\mathscr{F}(\alpha_2 X)$ are distinct so the $\beta - families$ of $\alpha_1 Y$ and $\alpha_2 Y$ are distinct as well. Therefore, Γ is one-to-one.

To show that Γ is onto, we first suppose that δY is a compactification of Y. By applying the above construction, there is a unique compactification δX of X whose $\beta - family$ is $\{h^{\leftarrow}(H) : H \in \mathscr{F}(\delta Y)\}$. But applying the construction to δX, we see there is a unique compactification $\delta_1 Y$ whose $\beta - family$ is $\{h(H) : H \in \mathscr{F}(\delta X)\} = \{h[h^{\leftarrow}[H] : H \in \mathscr{F}(\delta Y)\} = \{H : H \in \mathscr{F}(\delta Y)\}$ and $\Gamma(\delta X) = \delta_1 Y$. But compactifications are determined uniquely by their $\beta - families$. Thus $\delta_1 Y = \delta Y$ and Γ is onto.

Finally, we will use the following fact to complete the proof: A homomorphism between lattices is an isomorphism if and only if it is order-preserving by Theorem 5.2.10. Let $\alpha_1 X$ and $\alpha_2 X \in \mathscr{K}(X)$, where X is the space in the statement of the theorem. Then $\alpha_1 X \leq \alpha_2 X$ if and only if every set in $\mathscr{F}(\alpha_2 X)$ is contained in a set in $\mathscr{F}(\alpha_1 X)$ if and only if every member of $\{h(H) : H \in \mathscr{F}(\alpha_2 X)\} = \mathscr{F}(\alpha_2 Y)$ is contained in a member of $\{h(H) : H \in \mathscr{F}(\alpha_1 X)\} = \mathscr{F}(\alpha_2 Y)$ if and only if $\alpha_1 Y \leq \alpha_2 Y$ if and only if $\Gamma(\alpha_1 X) \leq \Gamma(\alpha_2 X)$. This completes the proof of Theorem 4.1.2

□

As an immediate consequence of 4.1.1 and 4.1.2, we have the following:

Corollary 4.1.3. *Suppose that X and Y are locally compact and non-compact. Then the lattices $\mathscr{K}(X)$ and $\mathscr{K}(Y)$ are isomorphic if and only if $\beta X \backslash X$ and $\beta Y \backslash Y$ are homeomorphic.*

4.2 Lattice Structure and The Automorphism Group

Definition 4.2.1. Suppose X is a space. We write $\mathscr{G}(X)$ to indicate the group of all autohomeomorphisms of X. (That is, the group of all automorphisms which continuous, and have a continuous inverse)

A quick check shows that the set of all autohomeomorphisms of a set X is in fact a subgroup of the group of all automorphisms of a set X (the composition of two autohomeomorphisms is an autohomeomorphism, and if h is an autohomeomorphism, then $h^{\leftarrow}$ exists and is an autohomeomorphism).

At this point, we provide results that relate the lattice structure of $\mathscr{K}(X)$ to the automorphism group of the lattice (and conversely when a group is given).

Corollary 4.2.2. *Let X be locally compact and non-compact, and let $\mathscr{A}(\mathscr{K}(X))$ indicate the automorphism group of the isomorphisms of $\mathscr{K}(X)$. If $|\beta X \backslash X| = 1$ or 2, then $\mathscr{A}(\mathscr{K}(X))$ is the group consisting of one element. If $|\beta X \backslash X| > 2$, then $\mathscr{A}(\mathscr{K}(X))$ is isomorphic to the group (with composition as the group operation) of all autohomeomorphisms of $\beta X \backslash X$.*

Proof. Observe that $\beta X \backslash X$ cannot be empty because this would imply that $\beta X = X$ so that X is compact. If $|\beta X \backslash X| = 1$, then by Theorem 4.1.1 part 1, $|\mathscr{K}(X)| = 1$ and the identity mapping is the only lattice isomorphism on $\mathscr{K}(X)$. Thus $\mathscr{A}(\mathscr{K}(X))$ has a single element. If $|\beta X \backslash X| = 2$, then $\mathscr{K}(X)$ includes just βX and the one-point compactification of X by part 1 of the proof of Theorem 4.1.1. Also, there is only one lattice isomorphism (the identity) on $\mathscr{K}(X)$. The reason is that βX must map to itself and therefore the one-point compactification must do the same.

Now suppose $|\beta X \backslash X| \neq 2$ and let Γ be an element of $\mathscr{A}(\mathscr{K}(X))$. By Theorem 4.1.1, there is a unique autohomeomorphism h of $\beta X \backslash X$ such that if $\Gamma(\alpha X) = \gamma X$ then $\mathscr{F}(\gamma X) = \{h(H) : H \in \mathscr{F}(\alpha X)\}$. We now define a function Φ from $\mathscr{A}(\mathscr{K}(X))$ to the group $\mathscr{G}(\beta X \backslash X)$ of all autohomeomorphisms of $\beta X \backslash X$ by $\Phi(\Gamma) = h$.

By construction, Φ is well-defined. To show that Φ is one-to-one, we will show that the kernel of Φ is the identity element of $\mathscr{A}(\mathscr{K}(X))$. Suppose $\Phi(\Gamma) = i$, the identity in $\mathscr{G}(\beta X\backslash X)$. Then if αX is a compactification of X and $\Gamma(\alpha X) = \gamma X$, $\mathscr{F}(\Gamma(\alpha X)) = \mathscr{F}(\gamma X) = \{i(H) : H \in \mathscr{F}(\alpha X)\} = \{H : H \in \mathscr{F}(\alpha X)\} = \mathscr{F}(\alpha X)$. That is, $\mathscr{F}(\gamma X) = \mathscr{F}(\alpha X)$ which implies that $\gamma X = \alpha X$ so that $\Gamma(\alpha X) = \gamma X = \alpha X$ and Γ is the identity in $\mathscr{A}(\mathscr{K}(X))$. By theorem 4.1.2, Φ is onto.

To show that Φ is a homomorphism, first suppose that $\Phi(\Gamma_1) = h_1$ and $\Phi(\Gamma_2) = h_2$. We need to show that $\Phi(\Gamma_1 \circ \Gamma_2) = \Phi(\Gamma_1) \circ \Phi(\Gamma_2)$. Suppose $\alpha X \in \mathscr{K}(X)$. Set $\Gamma_2(\alpha X) = \delta X$ and $\Gamma_1(\delta X) = \gamma X$ so that $\Gamma_1 \circ \Gamma_2(\alpha X) = \gamma X$. Then $\mathscr{F}(\delta X) = \mathscr{F}(\Gamma_2(\alpha X)) = \{h_2[H] : H \in \mathscr{F}(\alpha X)\}$. It follows that

$$\mathscr{F}(\gamma X) = \mathscr{F}(\Gamma_1 \circ \Gamma_2(\alpha X)) = \mathscr{F}(\Gamma_1(\delta X)) = \{h_1[H] : H \in \mathscr{F}(\delta X)\} = \{h_1[h_2[H]] : H \in \mathscr{F}(\alpha X)\}.$$

In particular, $\mathscr{F}(\Gamma_1 \circ \Gamma_2(\alpha X)) = \{h_1 \circ h_2(H) : H \in \mathscr{F}(\alpha X)\}$. By the definition of Φ, $\Phi(\Gamma_1 \circ \Gamma_2)) = h_1 \circ h_2 = \Phi(\Gamma_1) \circ \Phi(\Gamma_2)$.

□

The final result indicates that any group can be identified with the automorphism group of the lattice of compactifications of a certain locally compact space. The proof relies on two facts from papers preceding [5]. They are cited when used.

Corollary 4.2.3. *Given any group G, there exists a locally compact space X such that G is isomorphic to $\mathscr{A}(\mathscr{K}(X))$.*

Proof. Let G be a given group. If $|G| = 1$, take X to be any locally compact space such that $|\beta X\backslash X| = 1$. Then $\mathscr{A}(\mathscr{K}(X))$ is the trivial group which is isomorphic to G. Now suppose $|G| \neq 1$. By Theorem 8 in [3], there exists a compact connected Hausdorff space Y such that G is isomorphic to $\mathscr{G}(Y)$, the group of all autohomeomorphisms of Y. Since G has at least two elements, $\mathscr{G}(Y)$ must have at least two elements, so Y must have two elements. By problem 9K on page 138 of [4], Any Tychonoff space Y is homeomorphic to $\beta X\backslash X$ for a suitable space X.

We claim that X is locally compact. Since Y is homeomorphic to $\beta X\backslash X$, the image of Y under the homeomorphism is compact. Thus $\beta X\backslash X$ is compact. By Theorem 2.2.9 part b, $\beta X\backslash X$ is closed which implies that X is open in βX. But Corollary 8.14 of [6] states that a dense subset of a compact Hausdorff space is locally compact if and only if it is open. Since X is a dense subset of the compact Hausdorff space βX which is open, X is locally compact.

Also, Y is a connected Hausdorff space. We show Y has greater than two elements. We already noted that Y contains at least two elements. If Y contained two elements $\{a,b\}$, then $\{a\}$ and $\{b\}$ are non-empty disjoint open sets whose union is Y. This contradicts that Y is connected. Thus Y has greater than two elements. Since Y is homeomorphic to $\beta X \backslash X$, this means $\beta X \backslash X$ has greater than two elements as well. Since X is locally compact, and $\beta X \backslash X$ has greater than two elements, an application of Theorem 4.1.3 implies that $\mathscr{A}(\mathscr{K}(X))$ is isomorphic to $\mathscr{G}(\beta X \backslash X)$.

Since Y is homeomorphic to $\beta X \backslash X$, and G is isomorphic to $\mathscr{G}(Y)$, G is isomorphic to $\mathscr{G}(\beta X \backslash X)$. But $\mathscr{A}(\mathscr{K}(X))$ is isomorphic to $\mathscr{G}(\beta X \backslash X)$, so G is isomorphic to $\mathscr{A}(\mathscr{K}(X))$, completing the proof.

$\square$

References

[1] Richard E. Chandler and Gary D. Faulkner. Hausdorff compactifications: a retrospective. In *Handbook of the history of general topology, Vol. 2 (San Antonio, TX, 1993)*, volume 2 of *Hist. Topol.*, pages 631–667. Kluwer Acad. Publ., Dordrecht, 1998.

[2] B. A. Davey and H. A. Priestley. *Introduction to lattices and order*. Cambridge Mathematical Textbooks. Cambridge University Press, Cambridge, 1990.

[3] J. de Groot. Groups represented by homeomorphism groups i. *Mathematische Annalen*, 138(1):80–102, 1959.

[4] Leonard Gillman and Meyer Jerison. *Rings of continuous functions*. Springer-Verlag, New York-Heidelberg, 1976. Reprint of the 1960 edition, Graduate Texts in Mathematics, No. 43.

[5] Kenneth D. Magill, Jr. The lattice of compactifications of a locally compact space. *Proc. London Math. Soc. (3)*, 18:231–244, 1968.

[6] Jack Porter. *General Topology Notes*. Accessible at http://www.math.ku.edu/ porter/topology.pdf, 2013.

Chapter 5

Appendix

5.1 Function Results

First we provide some basic results about functions. The first two theorems are needed to prove the third theorem. The third theorem is used in the preliminaries section of this paper.

Theorem 5.1.1. *Suppose $g : A \to B$ and $f : B \to C$ are functions. Then $(f \circ g)^{\leftarrow}[D] = g^{\leftarrow}[f^{\leftarrow}[D]]$ for any $D \subseteq C$.*

Proof. ($\Rightarrow$) Let $a \in (f \circ g)^{\leftarrow}[D]$. Then there exists a $d \in D$ such that $(f \circ g)(a) = f(g(a)) = d$. Set $g(a) = b$. Then $f(b) = d$ means $b \in f^{\leftarrow}[D]$. But then $g(a) = b$ means $a \in g^{\leftarrow}[f(b)] \subseteq g^{\leftarrow}[f^{\leftarrow}[D]]$.

($\Leftarrow$) Let $a \in g^{\leftarrow}[f^{\leftarrow}[D]]$. Then there is a b $\in f^{\leftarrow}[D]$ such that $g(a) = b$. Since $b \in f^{\leftarrow}[D]$, there exists a $d \in D$ such that $f(b) = d$. Taken together, we have $f(g(a)) = d$ which implies $a \in (f \circ g)^{\leftarrow}[C]$.

$\square$

Theorem 5.1.2. *Suppose $g : A \to B$ and g is onto with $D \subseteq B$. Then $g[g^{\leftarrow}[D]] = D$.*

Proof. ($\Rightarrow$) Suppose $b \in g[g^{\leftarrow}[D]]$. Then there is an $a \in g^{\leftarrow}[D]$ such that $g(a) = b$. But $a \in g^{\leftarrow}[D] \to g(a) \in D \to b \in D$. Note how this direction does not require g to be onto.

($\Leftarrow$) Suppose $d \in D$. Since g is onto, there exists an a such that $g(a) = d$ so $a \in g^{\rightarrow}[D] \to g(x) = d$ for some $x \in g^{\leftarrow}[D]$ (in particular, when for x = a). In turn, $d \in g[g^{\leftarrow}[D]]$.

□

Theorem 5.1.3. *Suppose $f : X \to Z$, $g : X \to Y$, and $h : Y \to Z$ are functions so that $f = h \circ g$ with g onto. Then $h^{\leftarrow}[C] = g \circ f^{\leftarrow}[C] = g[f^{\leftarrow}[C]]$ where $C \subseteq Z$.*

Proof. Observe first that $f^{\leftarrow}[C] = (h \circ g)^{\leftarrow}[C] = g^{\leftarrow}[h^{\leftarrow}[C]]$, where the latter equality follows by Theorem 5.1.1. Thus $g[f^{\leftarrow}[C]] = g[g^{\leftarrow}[h^{\leftarrow}[C]]] = h^{\leftarrow}[C]$ by Theorem 5.1.2. □

5.2 Lattice Theory

The definitions of lattice and associated concepts are provided below. See [2] pages 3, 10, 27-29, 109, and 113 - 114 for original exposition.

Definition 5.2.1. Let P be a set. A **partial order** on P is a binary relation $\leq$ on P such that, for all $x, y, z \in P$,

(i) $x \leq x$ (Reflexive)

(ii) $x \leq y$ and $y \leq z$ imply $x = y$ (Symmetric)

(iii) $x \leq y$ and $y \leq z$ imply $x \leq z$ (Transitive)

Definition 5.2.2. Let L and K be partially ordered sets. A function $f : P \to Q$ is be

(i) **order-preserving** if $x \leq y$ in P implies $f(x) \leq f(y)$ in Q.

(ii) an **order-embedding** if $x \leq y$ in P if and only if $f(x) \leq f(y)$ in Q.

(iii) an **order-isomorphism** if it is an onto order-embedding.

Proposition 5.2.3. *An order-embedding is one-to-one.*

Proof. Let $f : P \to Q$ be an order-embedding. Suppose $f(x) = f(y)$. Then $f(x) \leq f(y)$ and $f(x) \geq f(y)$. Since f is an order-embedding, $x \leq y$ and $x \geq y$ as well. Since a partial order is antisymmetric, $x = y$ and f is one-to-one.

□

Since an order-embedding is one-to-one, an order-isomorphism is bijective, justifying the use of the word isomorphism.

Definition 5.2.4. Let P be an ordered set and $S \subseteq P$

(i) An element $x \in P$ is an **upper bound** of S if $s \leq x$ for all $s \in S$

(ii) An element $x \in P$ is a **lower bound** of S if $s \geq x$ for all $s \in S$.

(iii) An element x is the **least upper bound** of S, written $\sup\{S\}$, if

(a) x is an upper bound of S, and

(b) $x \leq y$ for all upper bounds y of S.

(iv) An element x is the **greatest lower bound** of S, written $\inf\{S\}$, if

(a) x is a lower bound of S, and

(b) $x \geq y$ for all upper bounds y of S.

We will use the notation $x \vee y$ for $\sup\{x,y\}$ and read as the join of x and y. In similar fashion, we use the notation $x \wedge y$ for $\inf\{x,y\}$ and read as the meet of x and y. We will use the notation $\bigvee S$ and $\bigwedge S$ to indicate $\sup\{S\}$ and $\inf\{S\}$. The symbols sup and inf are used to indicate supremum and infimum respectively. In general, there is no guarantee that any of the suprema or infima already noted exist. We are now prepared to provide the definition of a lattice.

Definition 5.2.5. Let P be a nonempty set partially ordered by $\leq$.

(i) If $x \vee y$ exists for all $x, y \in P$, then $(P, \leq)$ is called an **upper semilattice**. The notation $(P, \leq)$ is sometimes shortened to P.

(ii) If $x \wedge y$ exists for all $x, y \in P$, then $(P, \leq)$ is called a **lower semilattice**. The notation $(P, \leq)$ is sometimes shortened to P.

(iii) A lattice which is both an upper semilattice and a lower semilattice is called a **lattice**.

(iv) If $\bigvee S$ and $\bigwedge S$ exist for all $S \subseteq P$, then P is called a **complete lattice**. If one of $\bigvee S$ or $\bigwedge S$ exist, the lattice is called a **complete upper semilattice** or a **complete lower semilattice** respectively.

The following result is needed in the proof that for a Tychonoff space X, $\mathscr{K}(X)$ is a complete lattice if and only if X is locally compact.

Theorem 5.2.6. *A complete upper semilattice with a least element is a complete lattice.*

Proof. Let $(L, \leq)$ be a complete upper semilattice and set b to be the least element of L. Let $S \subseteq L$. We need to show that $\bigwedge S$ exists and $\bigwedge S \in L$. Let $T = \{a \in L : a \leq s, \text{ for all } s \in S\}$. Since $b \in T$, $T \neq \emptyset$. Since L is a complete upper semilattice, $\bigvee T$ exists and is a member of L. We claim that $\bigwedge S = \bigvee T$. Set $c = \bigvee T$. To verify that $c = \bigwedge S$, we need to show two things: (i) c is a lower bound for S, and (ii) if d is any other lower bound for S, then $d \leq c$. We show each in turn:

(i) Let $s \in S$. Then for all $a \in T$, $a \leq s$. This makes s an upper bound for T. Therefore, $\bigvee T \leq s$ since $\bigvee T$ is the least upper bound of T.

(ii) Suppose $d \leq s$ for all $s \in S$. Then $d \in T$. Thus $\bigvee T = c \geq d$ since $\bigvee T$ is an upper bound for T.

□

The following is used in the proof of Lemma 5.2.10.

Lemma 5.2.7. *Let L be a lattice and let $a, b \in L$. Then the following are equivalent:*

(i) $a \leq b$

(ii) $a \vee b = b$

(iii) $a \wedge b = a$

Proof. Suppose (i). By assumption and the reflexive property, b is an upper bound for $\{a, b\}$. Suppose z is also an upper bound for $\{a, b\}$. Then $b \leq z$. By definition, $b = a \wedge b$. Thus (i) implies (ii). By interchanging the word "upper" for "lower" and the symbol $\leq$ for $\geq$, the same proof shows that (i) implies (iii).

We now show that (ii) implies (i). Since b is an upper bound for {a,b}, $a \leq b$. Thus (ii) implies (i). Similarly, if we assume (iii), a is a lower bound for $\{a, b\}$ which means $a \leq b$. Thus (iii) implies (i).

Therefore, (i), (ii), and (iii) are equivalent.

□

Definition 5.2.8. Let L and K be lattices. A map $f : L \to K$ is said to be a **homomorphism** (or for emphasis, a **lattice homomorphism**) if f satisfies the following properties:

(a) $f(a \vee b) = f(a) \vee f(b)$ for all $a, b, \in L$

(b) $f(a \wedge b) = f(a) \wedge f(b)$ for all $a, b \in L$.

If f is bijective, we say that f is an **isomorphism** (or for emphasis, we say **lattice isomorphism**).

Proposition 5.2.9. *Let L and K be lattices and $f : L \to K$ a function. Then the following are equivalent:*

(i) f is order-preserving

(ii) For all $a, b \in L$, $f(a \vee b) \geq f(a) \vee f(b)$

(iii) For all $a, b \in L$, $f(a \wedge b) \leq f(a) \wedge f(b)$.

Proof. Suppose (i). We first show (ii). It is always true that $a \leq a \vee b$ and $b \leq a \vee b$. Since f is order-preserving, $f(a) \leq f(a \vee b)$ and $f(b) \leq f(a \vee b)$. That is, $f(a \vee b)$ is an upper bound for $\{f(a), f(b)\}$. By definition of supremum, $f(a \vee b) \geq f(a) \vee f(b)$. This verifies (ii). Now we show (iii). It is always true that $a \wedge b \leq a$ and $a \wedge b \leq b$. Since f is order-preserving, $f(a \wedge b) \leq f(a)$ and $f(a \wedge b) \leq f(b)$. That is, $f(a \wedge b)$ is a lower bound for $\{f(a), f(b)\}$. By definition of infimum, $f(a \wedge b) \leq f(a) \wedge f(b)$.

Now suppose (ii). We will show (i). Assume $a \leq b$ in L. By Lemma 5.2.7, $b = a \vee b$. Then we have $f(b) = f(a \vee b) \geq f(a) \vee f(b) \geq f(a)$ so that $f(a) \leq f(b)$. Finally, we suppose (iii) to show (i). Assume $a \leq b$ in L. By Lemma 5.2.7, $a = a \wedge b$. Then $f(a) = f(a \wedge b) \leq f(a) \wedge f(b) \leq f(b)$ so $f(a) \leq f(b)$ as needed.

We conclude that (i), (ii), and (iii) are equivalent.

□

Theorem 5.2.10. *Suppose L and K are lattices and $f : L \to K$ is a function between them. Then f is an order-isomorphism if and only if f is a lattice-isomorphism.*

Proof. First suppose f is a lattice isomorphism. Then

$$\begin{aligned} a \leq b \text{ in } L &\Leftrightarrow a \vee b = b && \text{(by Lemma 5.2.7)} \\ &\Leftrightarrow f(a \vee b) = f(b) && \text{(applying f and using that f is one-to-one)} \\ &\Leftrightarrow f(a) \vee f(b) = f(b) && \text{(since f is a lattice homomorphism)} \\ &\Leftrightarrow f(a) \leq f(b) && \text{(by Lemma 5.2.7} \end{aligned}$$

Now suppose f is an order-isomorphism. Let $a, b \in L$. We know from 5.2.9 that $f(a \vee b) \geq f(a) \vee f(b)$. We need to show that $f(a \vee b) \leq f(a) \vee f(b)$. Since f is onto, there exists a $c \in L$ such that $f(a) \vee f(b) = f(c)$. By the definition of supremum, $f(a) \leq f(c)$ and $f(b) \leq f(c)$. Since f is an order-isomorphism, $a \leq c$ and $b \leq c$. That is, c is an upper bound for $\{a, b\}$. By definition of supremum, $a \vee b \leq c$. Again using that f is an order-isomorphism, $f(a \vee b) \leq f(c) = f(a) \vee f(b)$. Thus $f(a \vee b) = f(a) \vee f(b)$.

We know from 5.2.9 that $f(a \wedge b) \leq f(a) \wedge f(b)$. By interchanging $\vee$ with $\wedge$, supremum with infimum, and $\leq$ with $\geq$ in the above proof, we conclude that $f(a \wedge b) \geq f(a) \wedge f(b)$. Thus $f(a \wedge b) = f(a) \wedge f(b)$ and f is a lattice isomorphism.

□